Notes on Stereographic Coordinates [draft]

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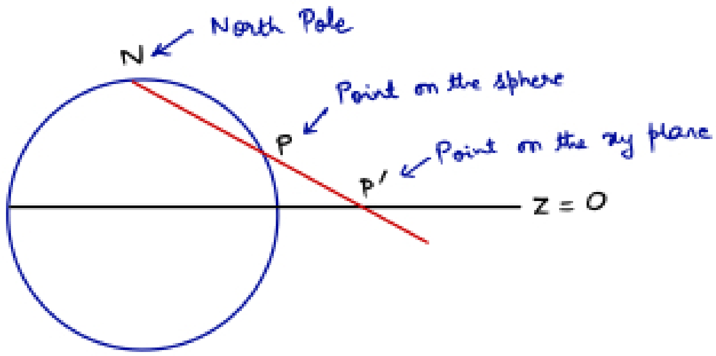
The purpose of this note is to explicitly list and derive some well-known relationships involving the stereographic projection of a point on the sphere to a plane passing through its equator. These relations are used heavily in theoretical physics, projective and algebraic geometry, and are often cast in different but equivalent ways.



Consider a sphere of radius r centered at the origin of a coordinate system. A point P on the sphere can be parametrized in terms of spherical coordinates (θ, ϕ) on the sphere (in our conventions, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$) as

$$(x_P, y_P, z_P) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \tag{1}$$

Let N denote the "north pole" of the sphere. By convention, we take $N \equiv (0, 0, r)$. The equation of a line joining N to P is

$$\frac{x-0}{a} = \frac{y-0}{b} = \frac{z-r}{c} = \lambda \tag{2}$$

where a, b, c are some nonzero real numbers (the "direction ratios" in the language of analytic geometry) and λ is a real parameter, such that $\lambda = 0$ corresponds to the north pole. This is a roundabout way of saying that any point on the line joining N to P has the parametric form

$$(x, y, z) = (a\lambda, b\lambda, r + c\lambda) \tag{3}$$

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For some value of λ, which we call λ_P, the point approaches the sphere. This is given by the requirement that $x_P^2 + y_P^2 + z_P^2 = r^2$ (r is the radius of the sphere). This determines λ_P through

$$(a^2 + b^2 + c^2)\lambda_P^2 + 2c\lambda_P r = 0 \tag{4}$$

Rejecting $\lambda_P = 0$ (for it corresponds to N, the north pole), we have

$$\lambda_P = -\frac{2rc}{a^2 + b^2 + c^2} \tag{5}$$

So, the point P has coordinates

$$(x_P, y_P, z_P) = \left(-\frac{2arc}{a^2 + b^2 + c^2}, -\frac{2brc}{a^2 + b^2 + c^2}, \frac{r(a^2 + b^2 - c^2)}{a^2 + b^2 + c^2}\right) \tag{6}$$

The line NP intersects the xy plane ($z = 0$) at a point P' for which $\lambda_{P'} = -\frac{r}{c}$. Hence, the coordinates of P' are

$$(x_{P'}, y_{P'}, z_{P'} = 0) = (X, Y, 0) = \left(-\frac{ar}{c}, -\frac{br}{c}, 0\right) \tag{7}$$

Combining these results, we get

$$\boxed{(x_P, y_P, z_P)_{\textsf{sphere,xyz}} = \left(\frac{2Xr^2}{X^2 + Y^2 + r^2}, \frac{2Yr^2}{X^2 + Y^2 + r^2}, \frac{r(X^2 + Y^2 - r^2)}{X^2 + Y^2 + r^2}\right)_{\textsf{plane,xyz}}} \tag{8}$$

where the subscripts on the coordinates emphasize that the coordinates of a point on the sphere are being related to the coordinates of a point on the plane. By solving for $X^2 + Y^2$ in terms of z_P and substituting the result in the expression for the other two coordinates (x_P and y_P), we get the inverse of this relationship, as

$$\boxed{(X, Y)_{\textsf{plane,xyz}} = \left(\frac{rx_P}{r - z_P}, \frac{ry_P}{r - z_P}\right)_{\textsf{sphere,xyz}}} \tag{9}$$

Substituting spherical coordinates (1) in (9), we get

$$\boxed{(X, Y)_{\textsf{plane,xyz}} = \left(r\cot\frac{\theta}{2}\cos\phi, r\cot\frac{\theta}{2}\sin\phi\right)_{\textsf{sphere,sph}}} \tag{10}$$

Expressing the coordinates on the plane itself in terms of polar coordinates (R, Θ), where

$$(X, Y)_{\textsf{plane,xyz}} = (R\cos\Theta, R\sin\Theta)_{\textsf{plane,polar}} \tag{11}$$

we have

$$\boxed{(R, \Theta)_{\textsf{plane,polar}} = \left(r\cot\frac{\theta}{2}, \phi\right)_{\textsf{sphere,sph}}} \tag{12}$$

or equivalently,

$$\boxed{(\theta, \phi)_{\textsf{sphere,sph}} = \left(2\tan^{-1}\frac{r}{R}, \Theta\right)_{\textsf{plane,polar}}} \tag{13}$$

As a perverse final step, suppose the point P on the sphere is parametrized in terms of *cylindrical* coordinates (ρ, φ, z), then using (1), we would have

$$\rho\cos\varphi = r\sin\theta\cos\phi \tag{14}$$
$$\rho\sin\varphi = r\sin\theta\sin\phi \tag{15}$$
$$z = r\cos\theta \tag{16}$$

These equations imply $\varphi = \phi$ and $\rho = r\sin\theta$. So, of course the obvious first inference is

$$\boxed{(\rho, \varphi, z)_{\textsf{sphere,cyl}} = (r\sin\theta, \phi, r\cos\theta)_{\textsf{sphere,sph}}} \tag{17}$$

and its inverse

$$\boxed{(\theta, \phi)_{\textsf{sphere,sph}} = \left(\sin^{-1}\frac{\rho}{r}, \varphi\right)_{\textsf{sphere,cyl}}} \tag{18}$$

The polar coordinates of the point P' on the *plane* are related to the cylindrical coordinates of the point P on the *sphere* via

$$\boxed{(\rho, \varphi, z)_{\textsf{sphere,cyl}} = \left(\frac{2r^2R}{r^2+R^2}, \Theta, \frac{r(R^2-r^2)}{R^2+r^2}\right)_{\textsf{plane,polar}}} \tag{19}$$

Having derived these results the "hard way" (or the "silly way", if you like) it is worth mentioning that there are a few slicker ways to get to these results. First of all, one could work with the unit sphere $(r = 1)$ and conformally rescale everything at the end. Secondly, one could directly use complex coordinates for points on the plane, and express points on the sphere in terms of these complex coordinates. The latter approach is in fact useful for applications of coset theory in the embedding formalism used to study, for instance, the spectrum of Kaluza-Klein modes on S^5.

The above stereographic projection above was from the north pole onto the equatorial plane. In order to get the corresponding expressions for the stereographic projection from the south pole, one needs to make the replacements $\theta \to \pi - \theta$ and $\phi \to -\phi$ (where θ and ϕ refer to the spherical coordinates of the point on the sphere).

References

[1] Wikipedia entry on Stereographic Coordinates, https://en.wikipedia.org/wiki/Stereographic_projection.

[2] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil. and E. Zaslow, eds. (2003). Mirror Symmetry (PDF). American Mathematical Society. ISBN 0-8218-2955-6.