The Anomalous Quantum Hall Effect

Vivek Saxena

Department of Physics & Astronomy
The State University of New York at Stony Brook

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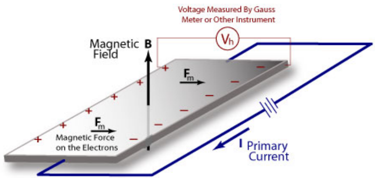
Classical Hall Effect



2D electron gas in a $\boldsymbol{B} = B\hat{e}_z$ field, with a longitudinal driving voltage (electric field: $E_x\hat{e}_x$). The Drude model + Newton's 2nd law yield

$$\frac{m\boldsymbol{v_d}}{\tau} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v_d}}{c} \times \boldsymbol{B}\right)$$

where $\boldsymbol{v}_d$ = drift velocity, τ = relaxation time (scattering time). In terms of Drude conductivity $\sigma = \frac{n_e e^2 \tau}{m}$ and the current density $\boldsymbol{j} = n_e e \boldsymbol{v}_d$,

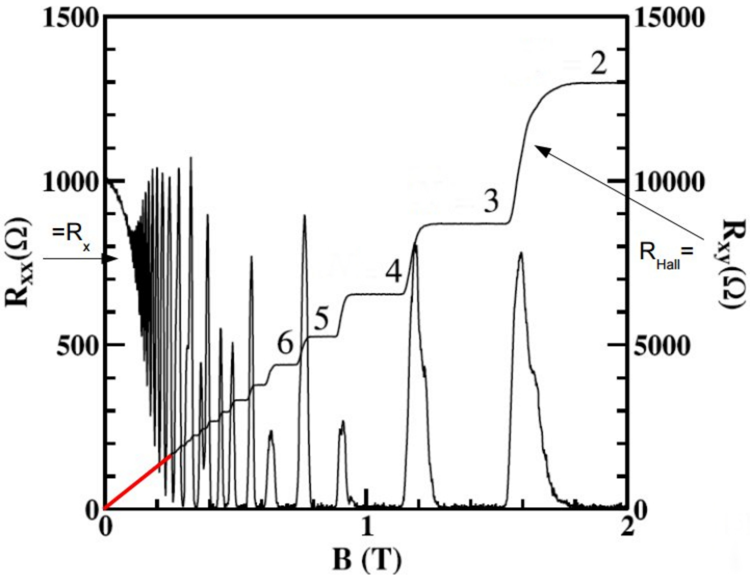
$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \sigma^{-1} & -\frac{B}{cen_e} \\ \frac{B}{cen_e} & \sigma^{-1} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} = -\rho_{xy} & \rho_{yy} = \rho_{xx} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

For $j_y = 0$, $E_x = \frac{B}{cen_e} j_x = \frac{B}{cen_e}\frac{I_x}{W}$ and $V_{Hall} = V_y = WE_y = \frac{B}{cen_e} I_x$.

So, $\boxed{R_{Hall} = \frac{B}{cen_e}}$. **Classical Hall resistance (or ρ_{xy}) $\propto B$.**

But...the Drude Model is *not* valid at large $\boldsymbol{B}$.

In fact...



So there is significant deviation from linearity in the variation of ρ_{xy} with the $\boldsymbol{B}$ field.

Outline

Quantum Hall Effect

- Quantization of Hall conductance, in 2D electron systems at **low T** and **strong B**.

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

- When σ_{xy} takes these values ("plateaus"), $\sigma_{xx} = 0$ **and** $\rho_{xx} = 0$.
- ν is called the *filling fraction*.
  - ⋆ $\nu \in \mathbb{Z}$: Integer Quantum Hall Effect (IQHE) [K. von Klitzing, G. Dorda, and M. Pepper (1980)].
  - ⋆ $\nu \in \mathbb{Q} \setminus \mathbb{Z}$: Fractional Quantum Hall Effect (FQHE) [D. Tsui, H. Stormer, and A. Gossard (1982)].
- The quantization of conductance is by now a well established metrological standard.
- Applications of QHE still far away because of extreme environments required to enter the QH regime.
- IQHE is fairly well understood, but we do not have a complete theory of the FQHE yet.
- This talk will be about a special kind of (I)QHE, called the Anomalous Quantum Hall Effect, or AQHE.

Why is the Hall conductance quantized?

A spinless electron of mass m and charge $-e$ in a magnetic field $\boldsymbol{B} = B\hat{e}_z$ is described by a one-particle Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\left[\left(-i\hbar\frac{\partial}{\partial x_1} - \frac{e}{c}A_1\right)^2 + \left(-i\hbar\frac{\partial}{\partial x_2} - \frac{e}{c}A_2\right)^2\right]$$

where $\boldsymbol{A}$ is such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, or $B = \epsilon_{ij}\partial_i A_j$. One choice is the symmetric gauge, $A_i = -\frac{1}{2}B\epsilon_{ij}x_j$. This is the famous Landau Level problem in non-relativistic quantum mechanics.

Solving the Schrödinger equation $\mathcal{H}\Psi_n = E_n\Psi_n$, the eigenvalues are found to be

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) \rightarrow \text{ called } \textbf{Landau levels}.$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency, and $n \in \{0, 1, \ldots\}$.

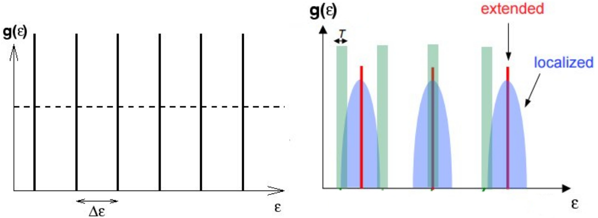
Why is the Hall conductance quantized?

For a 2D free electron gas, the density of states (DOS) is a constant

$$g(E) = \frac{m}{\pi\hbar^2}$$

But in the presence of a magnetic field, this splits into a Dirac delta chain

$$g(E) = \frac{m\omega_c}{\pi\hbar}\sum_n \delta\left[E - \hbar\omega_c\left(n + \frac{1}{2}\right)\right]$$

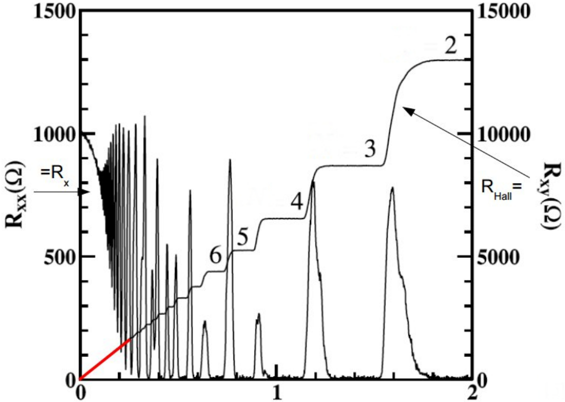


Due to impurities, the actual DOS broadens into 2 kinds of levels: extended and localized. Extended states are responsible for current conduction at low T. So, if the occupation of the extended states does not change, the current will not change.

Localized states

• Existence of localized states explains the plateaus of ρ_{xy}. As the electron density n is increased (or equivalently, $\boldsymbol{B}$ is decreased), localized states fill up without change in occupation of extended states; thus the Hall resistance ρ_{xy} does not change.

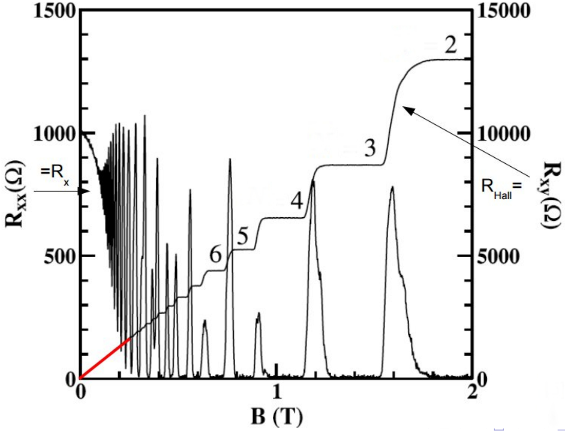
For such densities (or $\boldsymbol{B}$ values), ρ_{xy} is "constant", that is, on a plateau. Also, the longitudinal resistance $\rho_{xx} = 0$.



Extended states

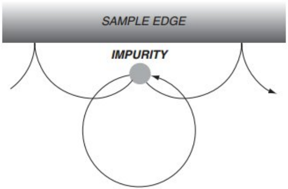
• Existence of extended states explains the peaks in ρ_{xx}. As the Fermi level E_F passes through a central maximum of an extended state, the longitudinal resistance ρ_{xx} becomes large, and the Hall resistance ρ_{xy} transitions from one plateau to the next.

Remember: only the states near E_F are relevant for conduction at low T.



Edge States

Real systems have physical boundaries (finite size). Edge states are QM versions of classical skipping orbits.



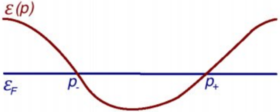
An impurity may momentarily disrupt forward propagation of an edge electron, but the Lorentz force is too strong and pushes it back to the boundary.

Magnetic length scale $l_B = \sqrt{\frac{\hbar c}{eB}}$ sets the scale above which electrons can be scattered.

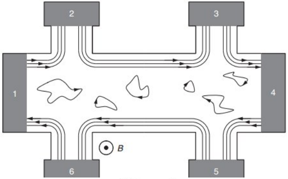
For $B \approx 4\,T$, the magnetic length $l_B \approx 10\,nm$.

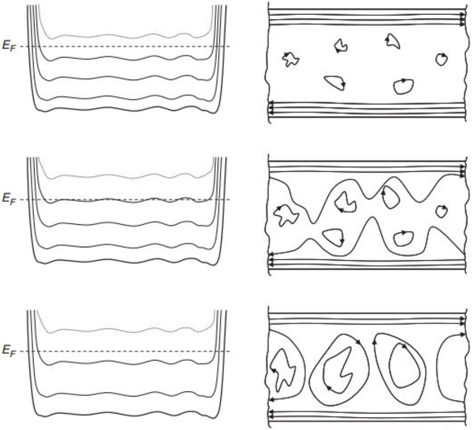
Scattering via impurities is strongly suppressed. Edge state transport is ballistic.

Bulk: conduction band edge is essentially flat, and almost independent of position. **Boundary**: conduction band edge rises well above E_F, intersecting it at two different points near the opposite edges of the sample. We thus have **two** *counterpropagating* edge states (1D channels) [**Chiral** edge modes.]



Chiral Edge Modes



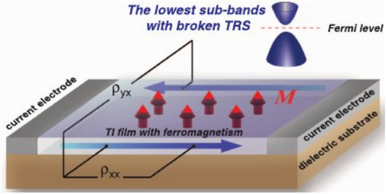


Magnetic field increasing from top to bottom. Increased localization is seen.

Outline

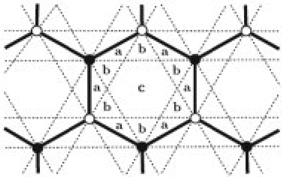
Anomalous Quantum Hall Effect

- Refers to the Quantum Hall Effect *without* an external magnetic field.
- Observed in thin films of Topological Insulators, specifically $(Cr_{0.12}Bi_{0.26}Sb_{0.62})_2Te_3$ as recently as 2013 (Chang *et al.*), 2014 (Kou *et al.*), conclusively established by 2015 (Bestwick *et al.* & Chang *et al.*).
- Temperatures still need to be low. But local ferromagnetic order replaces an external magnetic field.



- In 1988, Haldane proposed a lattice model with broken time reversal symmetry, but no external $\boldsymbol{B}$.

The Haldane Model



- σ^{xy} is odd under time reversal. Nonzero σ^{xy} means time reversal symmetry (TRS) is broken.

- The IQHE can arise in a 2D lattice system of spinless electrons in a periodic magnetic flux.

- Quantized Hall conductance is due to the band structure of electrons in the lattice instead of discrete Landau Levels.

- Bipartite honeycomb lattice (A (black) and B (white) sublattices) with 2 atoms per unit cell. Hopping terms t_1 for nearest-neighbor and t_2 for next-to-nearest neighbor hopping.

- On-site energy $+M$ on A-sites and $-M$ on B-sites break inversion symmetry.

- A periodic magnetic flux density $\boldsymbol{B}(\boldsymbol{r})$ normal to the plane, with zero flux through any unit cell. So, $\phi_a = -\phi_b$.
- Hopping terms t_2 acquire a phase $\phi = 2\pi(2\phi_a + \phi_b)/\phi_0$ where $\phi_0 = hc/e$.

The Haldane Model

The Hamiltonian in a basis of two-component Bloch spinor states $(c^{\dagger}_{k,A}, c^{\dagger}_{k,B})$ on the two sublattices (A (black) and B (white)) is

$$\mathcal{H}(\boldsymbol{k}) = \epsilon(k) + \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma} = \begin{pmatrix} \epsilon(k) + d_z(k) & d_x(k) - i d_y(k) \\ d_x + i d_y(k) & \epsilon(k) - d_z(k) \end{pmatrix}$$

where

$$\epsilon(k) = 2t_2 \cos\phi \sum_{i=1,2,3} \cos(\boldsymbol{k} \cdot \boldsymbol{b}_i)$$

$$d_x(k) = t_1 \sum_{i=1,2,3} \cos(\boldsymbol{k} \cdot \boldsymbol{a}_i), \qquad d_y(k) = t_1 \sum_{i=1,2,3} \sin(\boldsymbol{k} \cdot \boldsymbol{a}_i)$$

$$d_z(k) = M - 2t_2 \sin\phi \sum_{i=1,2,3} \sin(\boldsymbol{k} \cdot \boldsymbol{b}_i)$$

Here $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are displacements from a B site to its three nearest neighbor A sites such that $\hat{e}_z \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)$ is positive. Set of displacements to the six nearest neighbors on the same sublattice is $\{\pm \boldsymbol{b}_i\}$, where $\boldsymbol{b}_1 = \boldsymbol{a}_2 - \boldsymbol{a}_3$, $\boldsymbol{b}_2 = \boldsymbol{a}_3 - \boldsymbol{a}_1$, etc.

The Haldane Model: more details..

- Two bands, which touch only if $d_i(k) = 0$ for $i = 1, 2, 3$. This can happen only at a B.Z. corner $\boldsymbol{k}^0_\alpha$ for $M = \alpha 3\sqrt{3} t_2 \sin\phi$.
- If we take $|t_2/t_1| < 1/3$, the two bands never overlap, and are separated by a finite gap (unless they touch).
- Haldane then expanded the Hamiltonian in the neighborhood of a band extrema at the corners $\boldsymbol{k}^0_\alpha$ to linear order $\delta\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}^0_\alpha$.
- This gave, for the corners $(0, 2\pi/3, -2\pi/3)$ and $(0, -2\pi/3, 2\pi/3)$, respectively, the equations

$$\mathcal{H}_+ = v(\delta k_x \sigma_x - \delta k_y \sigma_y) + m_+ v^2 \sigma_z$$
$$\mathcal{H}_- = v(-\delta k_x \sigma_x - \delta k_y \sigma_y) + m_- v^2 \sigma_z$$

where $v = \frac{3}{2}\frac{t_1 a}{\hbar}$ and $m_\pm v^2 = M \mp 3\sqrt{3} t_2 \sin\phi$.

The Haldane Model: spectra..

The Hamiltonian $\mathcal{H}_\alpha$ is a $(1+2)$-D analog of the Dirac Hamiltonian. Its spectrum is,

$$\epsilon_{\alpha\pm}(\boldsymbol{k}) = \pm\sqrt{(\hbar c\boldsymbol{k})^2 + (m_\alpha c^2)^2} \text{ for } B_0 = 0$$

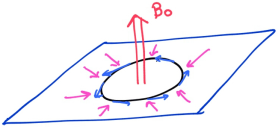
for $B_0 \neq 0$ we get "relativistic Landau Levels",

$$\begin{aligned}\epsilon_{\alpha n} &= \pm\sqrt{(m_\alpha c^2)^2 + n\hbar|eB_0|c^2} \text{ for } n \geq 1\\ \epsilon_{\alpha 0} &= \alpha m_\alpha c^2 \mathrm{sgn}(eB_0)\end{aligned}$$

- For $n \geq 1$, spectrum is $B_0 \to -B_0$ symmetric.
- The $n = 0$ "zero-mode" not symmetric under $B_0 \to -B_0$.
- In the TRS case, $t_2 \sin\phi = 0$, so $m_+ = m_-$. Sum of the spectra from the two B.Z. corners is invariant under $B_0 \to -B_0$. So, $\sigma^{xy} = 0$ by TRS.
- If $\mathrm{sgn}(m_+) = -\mathrm{sgn}(m_-)$, tune B_0 to make $m_+ = m_-$, and vary E_F so it lies in a gap. Compared to the TRS case, the system differs by complete filling of one Landau level.

The Haldane Model: conclusions

- Haldane concluded that at $T = 0$ and fixed chemical potential μ, applying a weak external $\boldsymbol{B}$ field to a system with m_+ and m_- having opposite signs results in a Hall conductance.
- Consider the system to be a Grand Canonical Ensemble (μ fixed by leads, and T fixed by refrigeration).



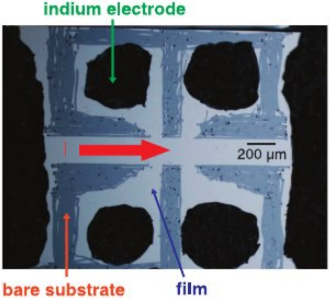
- Changing B_0 can draw in charge from leads: $\delta Q = \pm e \frac{B_0 A}{h/e} = \pm e^2 \frac{B_0 A}{h}$.
- Effective charge density $\sigma = \pm e^2 \frac{B_0}{h}$.
- Changing B_0 can give rise to transverse conductance $\sigma^{xy} = \left.\frac{\partial \sigma}{\partial B}\right|_{\mu,T}$.
- This is found to be $\sigma^{xy} = \nu \frac{e^2}{h}$, where $\nu = \frac{1}{2}(\text{sgn}(m_+) - \text{sgn}(m_-)) = \pm 1, 0$.

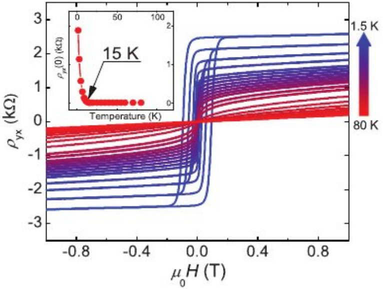
In modern parlance ν is called the first Chern number. It is possible to show the quantization of Hall conductance is a natural outcome of nontrivial band topology, by writing the conductance as an integral of a quantity called the Berry curvature.

Unfortunately I won't have time to describe this...

Outline

Experiment: $Cr_{0.15}(Bi_{0.1}Sb_{0.9})_{1.85}Te_3$

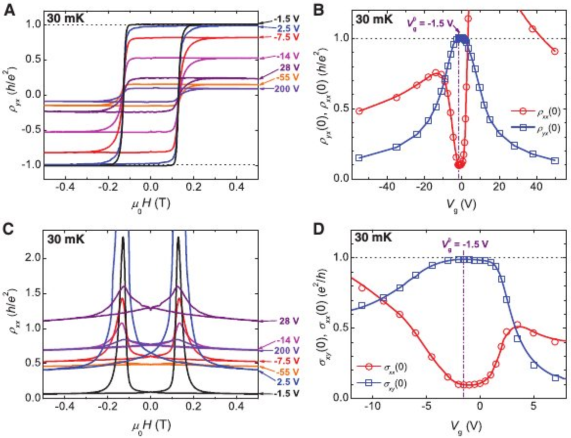




- Chang *et al.* [6] took a TI thin film, and measured its Hall characteristics at different temperatures.
- At high T, they reproduced (classical) Hall behavior: $\rho_{xy} \propto H$.
- At low T, they found a hysteresis loop, induced by ferromagnetic order in the film. The loop has high coercivity[1] (~ 970 Oersted at $T = 1.5\,K$) indicating long-range ferromagnetic order. The Curie temperature is found to be $\sim 15\,K$.

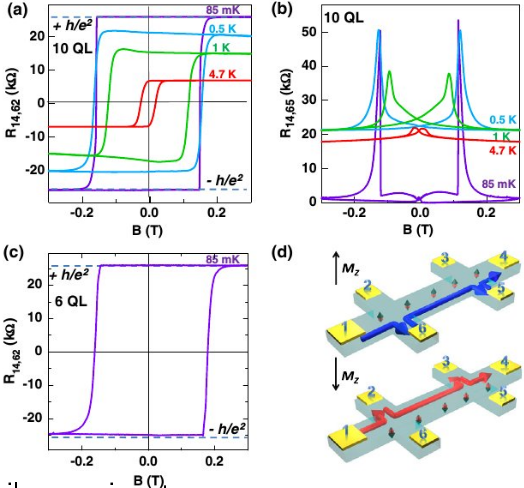
[1] Field required to eliminate residual magnetism upon reversal of the applied magnetic field.

Experiment:



- **A**: Shape and coercivity of ρ_{xy} hysteresis loops vary little with gate voltage V_g, due to robust ferromagnetic order. But height changes dramatically with V_g. We get quantization at $\pm h/e^2$ for $V_g \approx -1.5\,V$.
- **C**: Expected behavior for ρ_{xx}: magnetoresistance peaks at coercive fields.
- **B**: **Zero field** Hall resistance $\rho_{xy} = h/e^2$ for $V_g = -1.5\,V$ **[QAH!]**
- **D**: Zero field Hall conductivity $\sigma_{xy} = \frac{\rho_{xy}}{\rho_{xy}^2+\rho_{xx}^2}$ is on the e^2/h plateau.

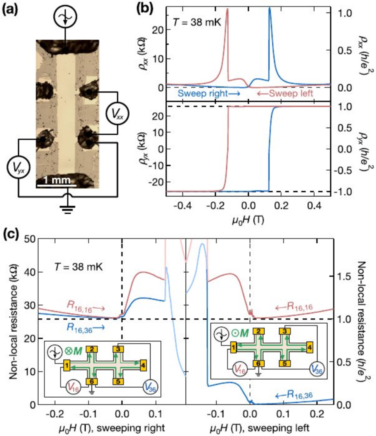
Experiment:



Kou *et al.* [7] did a similar experiment.

- **A**: Hysteresis curves $R_{xy} = R_{14,62}$ versus B.
- **B**: Magnetoresistance $R_{xx} = R_{14,65}$ peaks at coercive fields.
- $\boxed{\textbf{C}\text{: } \textbf{Zero field} \text{ Hall resistance } \rho_{xy} = h/e^2 \text{ at } T = 85\,mK.}$ **[QAH!]**
- **D**: The structure, with chiral edge states shown in color.

Experiment:



Bestwick *et al.* [9] repeated this, with careful attention to

1. Uneven spacing between voltage probes of Hall bar: mixing ρ_{xx} and ρ_{xy}.
2. Non-uniform device dimensions: estimation of precise dimensions from numerical solution of Laplace equation, and comparison to measured conductance.
3. Contact resistance.
4. Current loss to voltmeters.

Conclusions

- The Quantum Anomalous Hall Effect has been definitively observed in thin films of Cr-doped topological insulators, with broken time reversal symmetry.
- Local ferromagnetic order replaces the need for an external magnetic field.
- Temperatures still have to be low.
- Lattice model of Haldane "explains" certain aspects of QAHE, but a more concrete field theoretic model is still sought.
- Phenomenological approaches include including Chern Simons terms in a $(1+2)$-D field theory. These can explain quantization, but we do not yet have an ab-initio construction of such a model.

THANK YOU

I thank Prof. Lukasz Fidkowski for helpful discussions.

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The Haldane Model: more details..

- ⋆ The B.Z. is a hexagon rotated by $\pi/2$ w.r.t. to the Wigner-Seitz unit cell.
- ⋆ At its six corners of the form $(\boldsymbol{k}\cdot\boldsymbol{a}_1, \boldsymbol{k}\cdot\boldsymbol{a}_2, \boldsymbol{k}\cdot\boldsymbol{a}_3)$ is a permutation of $(0, 2\pi/3, -2\pi/3)$.
- ⋆ Two distinct corners $\boldsymbol{k}^0_\alpha$ are defined such that $\boldsymbol{k}^0_\alpha \cdot b_i = \alpha\frac{2\pi}{3}$ with $\alpha = \pm 1$.
- ⋆ The energy bands are obtained by diagonalizing the 2×2 matrix.
- There are two bands, which touch only if the three Pauli matrices have vanishing coefficients, that is $d_i(k) = 0$ for $i = 1, 2, 3$. This is possible only at the zone corner $\boldsymbol{k}^0_\alpha$ for $M = \alpha 3\sqrt{3}t_2 \sin\phi$.
- If we take $|t_2/t_1| < 1/3$, the two bands never overlap, and are separated by a finite gap (unless they touch).
- Haldane then expanded the Hamiltonian in the neighborhood of a band extrema at the corners $\boldsymbol{k}^0_\alpha$ to linear order $\delta\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}^0_\alpha$.
- This gave, for the corners $(0, 2\pi/3, -2\pi/3)$ and $(0, -2\pi/3, 2\pi/3)$, respectively, the equations

$$\mathcal{H}_+ = v(\delta k_x\sigma_x - \delta k_y\sigma_y) + m_+v^2\sigma_z$$
$$\mathcal{H}_- = v(-\delta k_x\sigma_x - \delta k_y\sigma_y) + m_-v^2\sigma_z$$

where $v = \frac{3}{2}\frac{t_1 a}{\hbar}$ and $m_\pm v^2 = M \mp 3\sqrt{3}t_2\sin\phi$.

The Haldane Model: still more details..

$$H_+ = v(\delta k_x \sigma_x - \delta k_y \sigma_y) + m_+ v^2 \sigma_z$$
$$H_- = v(-\delta k_x \sigma_x - \delta k_y \sigma_y) + m_- v^2 \sigma_z$$

The two Hamiltonians have different chirality when $m_\pm = 0$. To compare H_+ and H_- we can perform a transformation $(\sigma_x, \sigma_y, \sigma_z) \to (-\sigma_x, \sigma_y, -\sigma_z)$ on H_- and recognize $H_- = v(\delta k_x \sigma_x - \delta k_y \sigma_y) + \tilde{m}_- v^2 \sigma_z$ where $\tilde{m}_- = -m_- = -M - 3\sqrt{3} t_2 \sin\phi$.

The effective Hamiltonian near the corners can be written as $\mathcal{H}_\alpha = v(\delta k_x \sigma_x - \delta k_y \sigma_y) + \tilde{m}_\alpha v^2 \sigma_z$ where $\tilde{m}_\alpha = \alpha M - 3\sqrt{3} t_2 \sin\phi$.

The Haldane Model: spectra..

The Hamiltonian $\mathcal{H}_\alpha$ is a $(1+2)$-D analog of the Dirac Hamiltonian. Its spectrum is,

$$\epsilon_{\alpha\pm}(\boldsymbol{k}) = \pm\sqrt{(\hbar c\boldsymbol{k})^2 + (m_\alpha c^2)^2} \text{ for } B_0 = 0$$

for $B_0 \neq 0$ we get "relativistic Landau Levels",

$$\epsilon_{\alpha n} = \pm\sqrt{(m_\alpha c^2)^2 + n\hbar|eB_0|c^2} \text{ for } n \geq 1$$
$$\epsilon_{\alpha 0} = \alpha m_\alpha c^2 \mathsf{sgn}(eB_0)$$

- Every $n \geq 1$ level from the upper band (for $B_0 > 0$) is balanced by a level from the lower band.
- The $n = 0$ "zero-mode" energy is not symmetric under $B_0 \to -B_0$. It belongs to the upper band if $\alpha m_\alpha eB_0 > 0$ and to the lower band if $\alpha m_\alpha eB_0 < 0$.
- In the TRS case, $t_2 \sin\phi = 0$, so $m_+ = m_-$ and the *sum* of the Landau level spectra from the two distinct zone corners is invariant under $B_0 \to -B_0$. So, $\sigma^{xy} = 0$ by TRS. As the Hamiltonian is changed, σ^{xy} remains constant, if the Fermi level is in a gap.
- If m_+ and m_- have the opposite signs, we can vary B, make $m_+ = m_-$ equal and vary E_F to ensure it lies in a gap. Compared to the TRS case, the system differs by complete filling of one Landau level.

The Haldane Model: conclusions

Haldane concluded that at $T = 0$ and fixed chemical potential μ, applying a weak external $\boldsymbol{B}$ field to a system with m_+ and m_- having opposite signs results in a Hall conductance. This is given by $G_H = 2\pi^2 ec \left.\frac{\partial g(E_F)}{\partial B}\right|_{\mu,T}$.

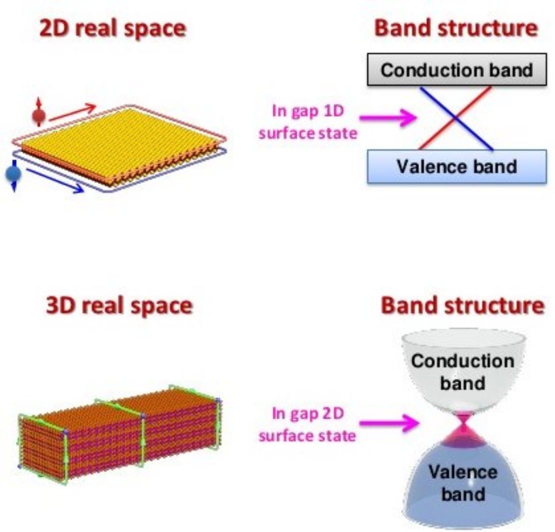
This relation is known as the Strada-Kubo relation and can be derived using linear response theory. Applying it to our density of states (Dirac chain, with weight $m\omega_c/\pi\hbar$), we get $G_H = 2\pi^2 ce \frac{\partial}{\partial B}\frac{m}{\pi\hbar}\frac{eB}{mc} = \frac{e^2}{h}$.

Including the contributions of the two chiral modes, this yields $\sigma^{xy} = \nu e^2/h$ where $\nu = \frac{1}{2}[\text{sgn}(m_-) - \text{sgn}(m_+)] = \pm 1$, or 0. Haldane's original calculation involved computing a field-dependent ground state charge density σ and using the thermodynamic (Strada) relation $\sigma^{xy} = \left.\frac{\partial \sigma}{\partial B}\right|_{\mu,T}$.

In modern parlance ν is called the first Chern number. It is possible to show the quantization of Hall conductance is a natural outcome of nontrivial band topology, by writing the conductance as an integral of a quantity called the Berry curvature.

Unfortunately I won't have time to describe this...

Band Structure of Topological Insulator



Q.H. Conductance and Berry Curvature

A two band model $H(k) = \epsilon(k) + \boldsymbol{d}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$ has eigenvalues $E \pm (\boldsymbol{k}) = \epsilon(\boldsymbol{k}) \pm d(\boldsymbol{k})$ where $d(\boldsymbol{k}) = \sqrt{\sum_{i=x,y,z} |d_i(\boldsymbol{k})|^2}$. Its eigenstates are $|\boldsymbol{k}, +\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}$ and $|\boldsymbol{k}, -\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$ where $\theta = \cos^{-1} \frac{d_z(\boldsymbol{k})}{d(\boldsymbol{k})}$ and $\phi = \tan^{-1} \frac{d_y(\boldsymbol{k})}{d_x(\boldsymbol{k})}$.

The conductivity $\sigma_{\alpha\beta}$ is given by $J_\alpha(\boldsymbol{r}, t) = \sum_\beta \sigma_{\alpha\beta}(\boldsymbol{q}, \omega) E_\beta \exp[i(\boldsymbol{q} \cdot \boldsymbol{r} - \omega t)]$.

The Kubo formula for the Hall conductance is $\boxed{\sigma_{\alpha\beta}(\boldsymbol{q}, \omega) = \frac{i}{\omega} \prod_{xy}(\boldsymbol{q}, \omega)}$ where $\prod_{xy}(\boldsymbol{q}, \omega)$ is the retarded correlation function of $J_x(\boldsymbol{q}, t)$ and $J_y(\boldsymbol{q}, t')$:

$$\boxed{\prod_{xy}(\boldsymbol{q}, \omega) = -\frac{i}{V} \int_{-\infty}^{\infty} dt\, \theta(t - t') e^{i\omega(t-t')} \langle \psi | [J_x(\boldsymbol{q}, t) J_y(\boldsymbol{q}, t')] | \psi \rangle}.$$

The DC conductivity is given by $\sigma_{\alpha\beta} = \lim_{\omega \to 0} \lim_{q \to 0} \sigma_{xy}(\boldsymbol{q}, \omega)$. Skipping steps (transform to imaginary time, take the zero frequency limit, etc.),

$$\boxed{\sigma_{xy} = -\frac{e^2}{h} \frac{1}{4\pi} \int dk_x dk_y \frac{((\partial_{k_x} \boldsymbol{d}(\boldsymbol{k}) \times \partial_{k_y} \boldsymbol{d}(\boldsymbol{k})) \cdot \boldsymbol{d}(\boldsymbol{k})}{d^3(\boldsymbol{k})}}$$