Stefano Cremonesi (Youtube) Lecture I

1. Monopoles, Dyons & Electro-Magnetic Duality
2. Basics of $\mathcal{N}=2$ SUSY
3. The Seiberg-Witten solution of the IR Physics of $\mathcal{N}=2$ SYM $\quad G=SU(2)$
4. Generalizations and the Seiberg Witten Curve from M5-branes

PART 1: MONOPOLES, DYONS & E-M DUALITY

Reviews: Coleman (Aspects of Symmetry)
Tong (TASI)
Harvey
Figueroa O'Farrell.

Maxwell EQNS

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E}=\rho & \vec{\nabla}\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=\vec{j} \\ \vec{\nabla}\cdot\vec{B}=0 & \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \end{array}} \longleftrightarrow$$

Tensor notation

$$\boxed{\begin{array}{l} \partial_\nu F^{\mu\nu}=j^\mu \\ \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}=0 \end{array}}$$

$$\updownarrow$$

Form notation

$$\boxed{\begin{array}{l} d*F=*j \\ dF=0 \end{array}}$$

There's an obvious E-M duality in the absence of sources. But Dirac's magnetic charge generalizes it to the situation when sources are present:

$$\boxed{\begin{array}{l} d*F=*j \\ dF=*K \end{array}}$$

$$\begin{pmatrix} *F \\ F \end{pmatrix} \longrightarrow M\begin{pmatrix} *F \\ F \end{pmatrix} \quad ; \quad \begin{pmatrix} j \\ K \end{pmatrix} \longrightarrow M\begin{pmatrix} j \\ K \end{pmatrix}$$

M is a general linear transformation.

In a quantum theory, charges are quantized, so there will be some restrictions on the matrix M.

Conventions followed are the same as in Tachikawa's review (2013; now a book).

U(1) GAUGE THEORY

$$A=A_\mu dx^\mu \quad ; \quad F=dA=\frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$$

U(1) group element: $g=e^{i\chi}$, $\chi\sim\chi+2\pi$

gauge transformation: $A\longmapsto A+ig^{-1}dg$ $\quad$ U(1) gauge transformation
$$=A-d\chi$$

$$\boxed{F=E^i\,dx^0\wedge dx^i+\frac{1}{2}\varepsilon_{ijk}B^i\,dx^j\wedge dx^k}$$

Useful decomposition!

$$\boxed{F = E^i dx^0 \wedge dx^i + \frac{1}{2}\varepsilon_{ijk} B^i dx^j \wedge dx^k}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \qquad *F_{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

$$\boxed{*F_{\mu\nu} = -B^i dx^0 \wedge dx^i + \frac{1}{2}\varepsilon_{ijk} E^i dx^j \wedge dx^k}$$

$$*F = \frac{1}{2}(*F)_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad ; \quad *F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}$$ Hodge dual

$$\boxed{\varepsilon^{0123} = -\varepsilon_{0123} = +1}$$

Maxwell's action

$$S_{Maxwell} = -\frac{1}{2e^2}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{e^2}\int F \wedge *F$$

Electric Charge

Field ϕ of $U(1)$ electric charge n. This means

$$\phi \mapsto g^n \phi = e^{inx}\phi$$

Covariant derivative

$$D_\mu \phi = (\partial_\mu + in A_\mu)\phi \mapsto g^n D_\mu \phi$$

Coupling electric charge n to A

$$S_{\text{electric charge}} = n\int_L A = n\int \delta_3(L) \wedge A$$

L ← worldline

closed 3-form with δ function support on the worldline L

$n \in \mathbb{Z}$: two ways to see this

1) $e^{inx} = e^{in(x+2\pi)} \Rightarrow e^{in2\pi} = 1 \Rightarrow n \in \mathbb{Z}$

2) Wick rotate to Euclidean space

weight in Euclidean path integral $\quad e^{-S^E_{\text{elec charge}}} = e^{-n\int_L A}$ $\quad$ Wilson line along L

Closed L + gauge invariance $\Rightarrow e^{in\oint A} = e^{in\oint (A - dx)} = e^{in\oint A}\, e^{-in\oint dx}$

$$\Rightarrow n \in \mathbb{Z}$$

So, the electric charge is quantized and the coupling to the charge has been introduced in such a way that the charge itself is an integer (and not an integer times a minimal charge).

EQUATIONS OF MOTION

$$S = S_{\text{Maxwell}} + S_{\text{electric charge}} = \frac{1}{e^2}\int F \wedge *F + n \int \delta_3(L) \wedge A$$

e.o.m: vary wrt A

$$\delta S = \frac{1}{e^2}\int \underset{\substack{\delta(dA) \\ = d(\delta A)}}{\underset{\shortparallel}{\delta F}} \wedge *F + \frac{1}{e^2}\int F \wedge *\delta F + n\int \delta_3(L) \wedge \delta A$$

Recall that $F \wedge *F = \frac{1}{4} F_{\mu\nu} F_{\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$

so $\delta F \wedge *F = F \wedge \delta * F$!

so,

$$\delta S = \frac{2}{e^2}\int d(\delta A) \wedge *F + n\int \delta_3(L) \wedge \delta A$$

Now, $d(\delta A \wedge *F) = \delta(dA) \wedge *F - \delta A \wedge d*F$

∴ Integrating by parts,

$$\delta S = \frac{2}{e^2}\int \delta A \wedge d*F - n\int \delta A \wedge \delta_3(L) + \frac{2}{e^2}\underbrace{\int d(\delta A \wedge *F)}_{\substack{\text{total derivative} \\ \rightarrow \text{ surface term} \rightarrow \text{drop}}}$$

↑ just a flip of the forms

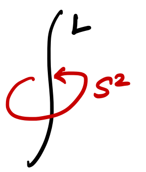
so, setting $\delta A = 0$, we get

$$\boxed{\frac{2}{e^2} d*F = n\,\delta_3(L)} \iff \boxed{\frac{4\pi}{e^2} d*F = 2\pi n\,\delta_3(L)}$$

Integrating the e.o.m over all space,

$$2\pi n = \int_{\mathbb{R}^3} \frac{4\pi}{e^2} d*F = \int_{S^2} \frac{4\pi}{e^2} *F \qquad [\text{Stokes' Theorem}]$$

$$= \frac{4\pi}{e^2}\oint_{S^2} \vec{E}\cdot d\vec{S} \qquad (\text{Electric flux}) \qquad [\text{Gauss's Law}]$$



2-sphere S^2 surrounding the worldline

So, the electric field generated by an electric charge n is

$$\boxed{\vec{E} = n\,\frac{e^2}{2}\,\frac{\hat{r}}{4\pi r^2}}$$

Magnetic Charge (DIRAC MONOPOLE)

1) Dirac '31
2) Wu-Yang '75
3) Alvarez '85

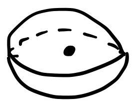
increasing level of sophistication

but also the # of patches that can be used to describe the Dirac monopole!

We follow the explanation due to Wu & Yang.
Requiring a magnetic charge (due to a monopole) is equivalent to requiring a nontrivial gauge bundle.

Locate the magnetic charge at the origin of $\mathbb{R}^3$.
Removing the origin, we have the space $\mathbb{R}^3 \setminus \{0\}$, which is CONTRACTIBLE to S^2.

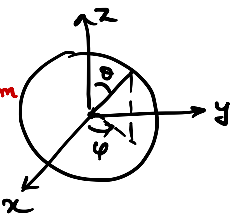
$$\mathbb{R}^3 \setminus \{0\} \sim S^2$$



The two-sphere is a manifold that requires two patches for its description.

$$\boxed{A^{\pm} = \frac{m}{2}(\pm 1 - \cos\theta)\, d\varphi}$$

gauge potential 1-form in usual spherical coordinates



Spherical coordinates

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Vielbein

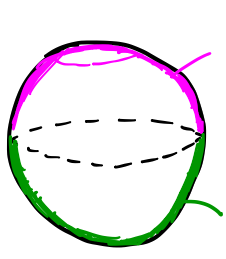
$$e^1 = dr$$
$$e^2 = r\,d\theta$$
$$e^3 = r\sin\theta\, d\varphi$$

The coordinates are not well defined everywhere.

- A^+ is regular everywhere except at $\theta = \pi$ (the south pole)
  That's because at the south pole, $A^+ = m\,d\varphi$ but φ is not well defined at the north or south pole.

  More rigorous explanation:
  $$A^+ = \frac{m}{2}(1-\cos\theta)\, d\varphi = \frac{m}{2}\frac{(1-\cos\theta)}{r\sin\theta} e^3 \quad \text{(diverges at } \theta = \pi)$$
  The vielbein 1-forms are well defined everywhere.
  There is no problem for A^+ near $\theta = 0$ because, we expand $\cos\theta$ & $\sin\theta$ near $\theta = 0$ to get
  $$A^+ \cong \frac{m}{2}\frac{\theta^2/2}{r\theta} e^3 = \frac{m}{4}\frac{\theta}{r} e^3 \quad \text{which goes to zero.}$$
  We could have seen that from $A^+ = \frac{m}{2}(1-\cos\theta)\,d\varphi$ itself but this argument makes it solid.

- Similarly, A^- is regular everywhere except at $\theta = 0$ (the North Pole).



$U^+ \cap U^-$ = equator

A^+ defined on $U^+ = S^2 \setminus \{SP\}$
A^- defined on $U^- = S^2 \setminus \{NP\}$

at the equator (overlap $U^+ \cap U^-$)
$$A^+|_{eq} - A^-|_{eq} = \frac{m}{2}d\varphi - \left(-\frac{m}{2}\right)d\varphi = m\,d\varphi$$
we want this to be a $U(1)$ gauge transformation
This happens for $g = \exp(im\varphi)$, with m being an integer! $(\varphi \to \varphi + 2\pi)$

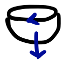
We want the gauge transformation $g = e^{im\varphi}$ to be single valued as φ goes from φ to $\varphi + 2\pi$. So, $\boxed{m \in \mathbb{Z}}$.

Magnetic Flux

$$F = dA = \frac{m}{2} \sin\theta \, d\theta \wedge d\varphi \qquad \text{using local coordinates}$$

So,
$$\int_{S^2} F = \frac{m}{2} \int_{S^2} \sin\theta \, d\theta \wedge d\varphi = \frac{m}{2} \times 4\pi = 2\pi m$$

More carefully,
$$\int_{S^2} F = \int dA^+ + \int dA^- = \oint_{eq.} A^+ - \oint_{eq.} A^- = 2\pi m$$

(using $A^+|_{eq} - A^-|_{eq} = m \, d\varphi$)

minus sign due to different direction of traversal of the equator (see figures below the integrals)

So the magnetic charge 'm' is the WINDING NUMBER of the U(1) gauge transformation around the circle.

As φ goes from 0 to 2π, the gauge transformation goes around the circle m times.

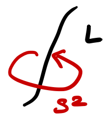
So,

flux of the magnetic field

$$\boxed{2\pi m = \int_{S^2} F = \oint_{S^2} \vec{B} \cdot d\vec{S}}$$

Bianchi Identity

$$\boxed{dF = 2\pi m \, \delta_3(L)}$$



Pointlike magnetic charge 'm' is often called a magnetic monopole or just monopole. It generates a magnetic field

$$\boxed{\vec{B} = 2\pi m \frac{\hat{r}}{4\pi r^2}}$$

- We have constructed a gauge field configuration for which dF is not everywhere zero, i.e. F is not globally closed. So we cannot write $F = dA$ globally.
- The Bianchi identity is satisfied in $\mathbb{R}^3 \setminus \{0\}$, where we <u>can</u> define a potential A such that $F = dA$, but the price is that A is defined patchwise. The gauge bundle is nontrivial, and a nontrivial gluing condition describes the magnetic charge.
- Dirac's original construction did not use patches but had the Dirac string.

Lecture II — DYNAMICS OF N=2 SUSY GAUGE THEORIES

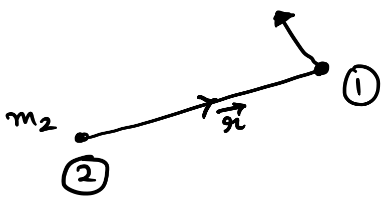
Recall :

$$2\pi n = \int_{S^2} \frac{4\pi}{e^2} * F = \oint_{S^2} \frac{4\pi}{e^2} \vec{E}\cdot d\vec{S} \qquad \text{electric charge}$$

$$2\pi m = \int_{S^2} F = \oint_{S^2} \vec{B}\cdot d\vec{S} \qquad \text{magnetic charge}$$

DIRAC - SCHWINGER - ZWANZIGER QUANTIZATION

Consider electric charge n_1 ① in the field of a magnetic charge m_2 ②.



$$\boxed{\frac{d}{dt}\vec{P}_1 = n_1 \dot{\vec{r}} \times \vec{B}}$$

Lorentz force on electric charge due to magnetic field $\vec{B}$ generated by ②

$\vec{L}_1$ = orbital angular momentum of electric charge about magnetic charge

$$\frac{d}{dt}\vec{L}_1 = \frac{d}{dt}(\vec{r}\times\vec{p}_1)$$

$$= \underbrace{\dot{\vec{r}}\times\vec{p}_1}_{\substack{\text{zero} \\ \text{as } \vec{p}_1 \propto \dot{\vec{r}}}} + \vec{r}\times\frac{d\vec{p}_1}{dt}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \quad -(*)$$

$$= \vec{r}\times(n_1\dot{\vec{r}}\times\vec{B})$$

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{r\frac{d\vec{r}}{dt} - \vec{r}\frac{dr}{dt}}{r^2} = \frac{1}{r}\dot{\vec{r}} - \hat{r}\frac{dr}{dt} \qquad (\#)$$

$$= n_1\vec{r}\times\left(\dot{\vec{r}}\times\frac{m_2}{2r^3}\vec{r}\right) \qquad \text{use } (*)$$

$$= \frac{n_1 m_2}{2r^3}\left[\dot{\vec{r}}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\dot{\vec{r}})\right]$$

$$= \frac{n_1 m_2}{2r}\left[\dot{\vec{r}} - \underbrace{\hat{r}(\hat{r}\cdot\dot{\vec{r}})}_{\text{radial component of } \dot{\vec{r}}}\right]$$

$$= \frac{n_1 m_2}{2r}\left[\dot{\vec{r}} - \hat{r}(\hat{r}\cdot[\hat{r}\dot{r} + \ldots])\right]$$

$$= \frac{n_1 m_2}{2r}\left[\dot{\vec{r}} - \hat{r}\frac{dr}{dt}\right] = \frac{d}{dt}\left(\frac{n_1 m_2}{2}\frac{\vec{r}}{r}\right) \quad (\text{see } \#)$$

Slicker way to get to this...

$$\frac{d}{dt}\vec{L}_1 = \frac{n_1 m_2}{2r^3}\left[\vec{r}\times(\dot{\vec{r}}\times\vec{r})\right]$$

$\hat{r}$, $\hat{\theta}$, $\hat{z}$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\dot{\vec{r}}\times\vec{r} = r^2\dot{\theta}(\hat{\theta}\times\hat{r}) = -r^2\dot{\theta}\hat{z}$$

$$\vec{r}\times(\dot{\vec{r}}\times\vec{r}) = r^3\dot{\theta}\hat{\theta}$$

$$\Rightarrow \frac{\vec{r}\times(\dot{\vec{r}}\times\vec{r})}{r^3} = \dot{\theta}\hat{\theta} = (\dot{\vec{r}} - \dot{r}\hat{r})/r = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

(still use #)

$$\Rightarrow \boxed{\frac{d\vec{L}_1}{dt} = \frac{d}{dt}\left(\frac{n_1 m_2}{2}\frac{\vec{r}}{r}\right)}$$

∴ The orbital angular momentum of particle 1 w.r.t. particle 2 is not conserved in time.

But there is a conserved angular momentum given by

$$\boxed{\frac{d}{dt}\vec{J}_{tot} = 0}, \quad \boxed{\vec{J}_{tot} = \vec{L} - \underbrace{\frac{n_1 m_2}{2}\frac{\vec{r}}{r}}_{\vec{J}_{em}}} \qquad \text{conserved}$$

Quantization of angular momentum requires

$$|\vec{J}_{em}| \in \frac{1}{2}\mathbb{Z}$$

(in units where $\hbar = 1$)

$$\therefore \quad \boxed{n_1 m_2 \in \mathbb{Z}}$$

This may seem silly as we began with integers n_1, m_2.

The point here is that if we normalize the charges in such a way that

$$n \rightsquigarrow q \qquad \text{i.e. } q = ne$$

$$m \rightsquigarrow g$$

we would then find that (reinstating factors of $\hbar$).

$$\boxed{\frac{q_1 g_2}{2\pi\hbar} \in \mathbb{Z}}$$

10:05

Exercise: Show that $\boxed{\vec{J}_{em} = \int d^3\vec{r}\; \vec{r} \times \underbrace{(\vec{E}\times\vec{B})}_{\substack{\text{Poynting vector} \\ \text{(momentum density} \\ \text{of EM field)}}}}$.

Solution: Suppose the magnetic charge is at the origin $\vec{r}' = 0$. So, its magnetic field at $\vec{r}'$

$$\vec{B} = \frac{m}{2}\frac{\hat{r}'}{r'^2} = \frac{m}{2}\frac{\vec{r}'}{r'^3}$$

Suppose the electric charge is at $\vec{r}' = \vec{r}_0$. Its electric field at $\vec{r}'$ is

$$\vec{E} = \frac{ne^2}{8\pi}\frac{\vec{r}'-\vec{r}_0}{|\vec{r}'-\vec{r}_0|^3}$$

For simplicity, take $\vec{r}_0 = r_0\hat{z}$, and $\vec{r}' = (r'\sin\theta\cos\phi, r'\sin\theta\sin\phi, r'\cos\theta)$ where $r' \in [0,\infty)$, $\theta \in [0,\pi]$, $\phi \in [0, 2\pi)$ as usual. Then,

$$\vec{E}\times\vec{B} = \frac{nme^2}{16\pi}\frac{1}{r'^3}\frac{1}{|\vec{r}'-\vec{r}_0|^3}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r'\sin\theta\cos\phi & r'\sin\theta\sin\phi & r'\cos\theta - r_0 \\ r'\sin\theta\cos\phi & r'\sin\theta\sin\phi & r'\cos\theta \end{vmatrix}$$

$$= \frac{nme^2}{16\pi}\frac{1}{r'^3}\frac{1}{|\vec{r}'-\vec{r}_0|^3}\Big\{\hat{x}\big[r'^2\sin\theta\cos\theta\sin\phi - (r'\cos\theta - r_0)(r'\sin\theta\sin\phi)\big]$$

$$- \hat{y}\big[(r'\sin\theta\cos\phi)(r'\cos\theta) - (r'\cos\theta - r_0)(r'\sin\theta\cos\phi)\big]\Big\}$$

$$\{\,\}_{\hat{x}} = r'^2\sin\theta\cancel{\cos\theta}\sin\phi - r'^2\sin\theta\cancel{\cos\theta}\sin\phi + r_0 r'\sin\theta\sin\phi$$

$$= r_0 r'\sin\theta\sin\phi$$

$$\{\,\}_y = r'^2\sin\theta\cos\theta\cos\phi - r_0 r'\sin\theta\cos\phi - r'^2\sin\theta\cos\theta\cos\phi$$

$$= -r_0 r'\sin\theta\cos\phi$$

so $\{\,\} = r_0 r'\sin\theta(\hat{x}\sin\phi - \hat{y}\cos\phi) = -r_0 r'\sin\theta\,\hat{e}_\phi$

so $\vec{r}'\times\{\,\} = +r_0 r'^2\sin\theta\,\hat{e}_\theta$

$\hat{r}$ $\hat{\phi}$ $\hat{\theta}$

$$|\vec{r}'-\vec{r}_0|^2 = r'^2\sin^2\theta\cos^2\phi + r'^2\sin^2\theta\sin^2\phi + (r'\cos\theta - r_0)^2$$

$$= r'^2\sin^2\theta + r'^2\cos^2\theta + r_0^2 - 2r'r_0\cos\theta$$

$$= r'^2 + r_0^2 - 2r'r_0\cos\theta$$

$$\Rightarrow |\vec{r}'-\vec{r}_0|^3 = (r'^2 + r_0^2 - 2r'r_0\cos\theta)^{3/2}$$

so,

$$\vec{r}'\times(\vec{E}\times\vec{B}) = \frac{nme^2}{16\pi}\frac{1}{r'^3}\frac{1}{(r'^2+r_0^2-2r'r_0\cos\theta)^{3/2}} \times r_0 r'^2 \sin\theta\, \hat{e}_\theta$$

$$= \frac{nme^2}{16\pi}\frac{r_0\sin\theta}{r'(r'^2+r_0^2-2r'r_0\cos\theta)^{3/2}}\hat{e}_\theta$$

now, $\hat{e}_\theta = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

(Source: http://plaza.obu.edu/corneliusk/mp/suv.pdf)

so,

$$\int d^3r'\,\vec{r}'\times(\vec{E}\times\vec{B}) = \frac{nme^2}{16\pi}(\hat{x}I_1 + \hat{y}I_2 + \hat{z}I_3)$$

where

$$I_1 = \int d^3r' \frac{r_0\sin\theta\cos\theta\cos\phi}{r'(r'^2+r_0^2-2r'r_0\cos\theta)^{3/2}} = 0 \quad \text{as} \quad \int_0^{2\pi} d\phi\,\cos\phi = 0$$

$$I_2 = \int d^3r' \frac{r_0\sin\theta\cos\theta\sin\phi}{r'(r'^2+r_0^2-2r'r_0\cos\theta)^{3/2}} = 0 \quad \text{as} \quad \int_0^{2\pi} d\phi\,\sin\phi = 0$$

$$I_3 = -\int d^3r' \frac{r_0\sin^2\theta}{r'(r'^2+r_0^2-2r'r_0\cos\theta)^{3/2}} \leftarrow \text{only integral to be evaluated}$$

$d^3r' = r'^2\sin^2\theta\, d\theta\, d\phi$

DYONS : Particles with both electric and magnetic charge.

Dyon ① charge $(\overset{el}{n_1}, \overset{mag}{m_1})$

Dyon ② charge (n_2, m_2)

▶ The quantized combination now is

$$\boxed{(n_1 m_2 - n_2 m_1) \in \mathbb{Z}}$$

→ Dirac Schwinger – Zwanziger Pairing of (n_1, m_1) & (n_2, m_2)

This follows from the fact that J_{em} is antisymmetric in $\vec{E}$ & $\vec{B}$.

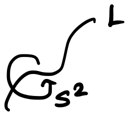
THETA ANGLE & WITTEN EFFECT

$$S_{em} = S_{Maxwell} + \underbrace{S_\theta}_{\text{Topological term}} = \frac{1}{e^2}\int F \wedge * F - \frac{\theta}{8\pi^2}\int F \wedge F \qquad \text{[Form notation]}$$

$$= -\frac{1}{2e^2}\int d^4x \, F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{32\pi^2}\int d^4x \, F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \qquad \text{[Tensor notation]}$$

Consider this action with a source term for an electric charge.

$$S = S_{em} + S_{el.ch.n} \rightsquigarrow \text{e.o.m} : \quad d\left[\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right] = 2\pi n \, \delta_3(L)$$



We want to derive the formula for the electric charge.

The first thing to notice here is that in vacuum where there are no electric charges, the conserved quantity is no longer $\frac{4\pi}{e^2} * F$, but rather $\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F$.

Integrating the equation of motion, we get a new formula for the charge

$$\boxed{\underset{\text{U(1) charge}}{2\pi n} = \int_{S^2}\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right) = \oint_{S^2}\left(\frac{4\pi}{e^2}\vec{E} - \frac{\theta}{2\pi}\vec{B}\right) \cdot d\vec{S}}$$

now we call it a U(1) charge

Note that we still have the second formula

$$\boxed{2\pi m = \int_{S^2} F = \oint_{S^2} \vec{B} \cdot d\vec{S}}$$

(Consequence of Bianchi Identity)

for the magnetic field. It is not affected by the θ-angle term. The action was not used to write this formula.

So, combining these equations, we get

$$\boxed{\int_{S^2} \frac{4\pi}{e^2} * F = \underbrace{\oint_{S^2} \frac{4\pi}{e^2} \vec{E} \cdot d\vec{S}}_{\text{"electric charge"}} = 2\pi\Big(\underbrace{n}_{\text{U(1) charge}} + \frac{\theta}{2\pi}\underbrace{m}_{\text{"magnetic charge"}}\Big)}$$

This phenomenon that the electric charge is modified from the U(1) charge by a term involving the product of the θ-angle and the magnetic charge is called the WITTEN EFFECT.

NOTE: From now on, n will be referred to as the "electric charge", even though it is not the flux of the electric field $\vec{E}$ but of $\frac{4\pi}{e^2}\vec{E} - \frac{\theta}{2\pi}\vec{B}$.

m is still called the magnetic charge, as before.

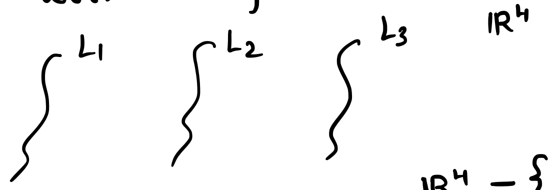
- The topological term added to the action measures the instanton number. The normalization chosen here is such that

$$\frac{1}{8\pi^2}\int F\wedge F \in \mathbb{Z}$$

- Θ is actually an angle, called the 't-Hooft Θ-angle.
- There are no instantons for $U(1)$. But non-abelian theories do have abelian subgroups. In the IR limit, these theories can be described by abelian subgroups, and the UV-nonabelian theory can induce an instanton number for the abelian low energy theory.

The second reason for the Θ-term is that we can have both electric and magnetic charges which are treated as external sources:

e.g. $\mathbb{R}^4$ with worldlines of external classical sources



abelian gauge theory lives on $\mathbb{R}^4 - \{L_1, L_2, L_3, \ldots\}$

on such spacetimes, one can have a nontrivial instanton number even if the gauge group is $U(1)$.

The instanton number in such cases will be determined by the electric and magnetic charges.

ELECTRIC-MAGNETIC DUALITY & $SL(2,\mathbb{Z})$

Let us first consider $\Theta = 0$.

$$2\pi n = \oint_{S^2} \frac{4\pi}{e^2}\vec{E}\cdot d\vec{S} \qquad\qquad 2\pi n\,\delta_3(L) = \frac{4\pi}{e^2}\, d*F$$

$$2\pi m = \oint_{S^2} \vec{B}\cdot d\vec{S} \qquad\qquad 2\pi m\,\delta_3(L) = dF$$

The integrated equations and the local e.o.m's are invariant under Electric-Magnetic Duality (S-TRANSFORMATION).

$$F = (\vec{E}, \vec{B}) \longmapsto \underset{\text{Dual}}{F_D} = (\vec{E}_D, \vec{B}_D) = -\frac{4\pi}{e^2}*F$$

$$= \frac{4\pi}{e^2}(\vec{B}, -\vec{E})$$

$$(n, m) \longmapsto (n_D, m_D) = (m, -n)$$

$$\frac{4\pi}{e^2} \longmapsto \frac{4\pi}{e_D^2} = \frac{e^2}{4\pi}$$

equations of motion & Bianchi Identity are invariant under these transformations

e.g.:

$$2\pi n = \oint_{S^2}\frac{4\pi}{e^2}\vec{E}\cdot d\vec{S}$$

$$\parallel$$

$$-2\pi m_D = -\oint_{S^2}\vec{B}_D\cdot d\vec{S}$$

It is worth emphasizing that the Electric-Magnetic Duality or S-Transformation is not a symmetry of the theory. It merely gives a dual description of the original theory. But it also acts on the coupling and is hence not a symmetry transformation.

S-Transformation for path integral.

$$Z = \int \mathcal{D}A \; e^{iS[A]}$$

$$S[A] = \frac{1}{e^2}\int dA \wedge * dA$$

only depends on curvature (for gauge invariance)

But Z depends on A

so, Z can be written as

$$Z = \int \mathcal{D}A_D \int \mathcal{D}F \; e^{i\int F\wedge *F + \frac{1}{2\pi}\int A_D \wedge F}$$

crucial * factor

where F is an unconstrained 2-form

A_D = some Lagrange multiplier : its role is to impose the Bianchi identity $dF = 0$

(we need this because we're working with F which is unconstrained, and not A)

1) Path integral $\int \mathcal{D}A_D$ first $\longrightarrow \delta(dF=0)$

(lets not worry about manifolds other than $\mathbb{R}^4$)

on $\mathbb{R}^4$, this means $F = dA$

so, doing this path integral first, we get the Bianchi identity and hence recover the original formulation.

2) Path integral $\int \mathcal{D}F$ first. quadratic action in $F \Rightarrow$ Gaussian integral.

The exponent has a quadratic action in F

$$\hookrightarrow \frac{1}{e^2}\int F\wedge *F + \frac{1}{2\pi}\int F_D \wedge F$$

(where $F_D = dA_D$)

after integrating $\int A_D \wedge F$ by parts

$$= \frac{1}{e^2}\int \left(F - \frac{e^2}{4\pi} * F_D\right)\wedge * \left(F - \frac{e^2}{4\pi} * F_D\right) + \left(\frac{e^2}{4\pi}\right)^2 F_D \wedge * F_D$$

match

Gaussian integral over F $\Rightarrow \; F = \frac{e^2}{4\pi} * \underbrace{F_D}_{= dA_D}$

so, up to a normalization factor,

$$\boxed{Z = \int \mathcal{D}A_D \; e^{\frac{i}{e_D^2}\int dA_D \wedge * dA_D}}$$

where $e_D^2 = \frac{(4\pi)^2}{e^2}$

This is what we earlier got by exchanging equations of motion with the Bianchi identity.

* One crucial factor we put above was $\frac{1}{2\pi}$.

NORMALIZATION : magnetic source $m \longrightarrow \frac{1}{2\pi}\int A_D \wedge (dF - 2\pi m \, \delta_3(L))$

we have an extra term $\frac{1}{2\pi}\int 2\pi m \, \delta_3 \wedge A_D = m \int \delta_3 \wedge A_D$

The 2π was there because now the magnetic charge appears as a dual electric charge coupled to the dual potential with the right integer coefficient.

$\longrightarrow 2\pi$ factor ensures proper charge quantization.

Exercise: Repeat (at least one of) the 2 arguments when $\theta \neq 0$.

Show that under the S-transformation,

$$S: \quad F \longmapsto F_D = -\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right)$$

$$(n,m) \longmapsto (n_D, m_D) = (m, -n)$$

Complexified coupling $\quad \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \longmapsto \tau_D = \frac{\theta_D}{2\pi} + i\frac{4\pi}{e^2} = -\frac{1}{\tau}$

These 3 transformations can be written in a compact way.

$$\begin{pmatrix} -\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right) \\ F \end{pmatrix} \longmapsto \begin{pmatrix} -\left(\frac{4\pi}{e_D^2} * F_D - \frac{\theta_D}{2\pi} F_D\right) \\ F_D \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix}$$

Also, $\displaystyle\int_{S^2} \underbrace{\qquad}$ $= \begin{pmatrix} -n \\ m \end{pmatrix}$

- Note that under the S-transformation, what transforms simply is the field strength. There is no simple duality transformation for the potentials.

- There is an extra transformation that we can consider, due to the θ-angle.

- θ is called theta "angle" because $\theta \sim \theta + 2\pi$ in the Euclidean path integral.

$$e^{i\theta \frac{1}{2\pi}\int \left(\frac{F}{2\pi}\right)\wedge\left(\frac{F}{2\pi}\right)} \quad \in \mathbb{Z} \text{ (for spin manifolds)}$$

$\longrightarrow$ instanton number

Caveat: on spin manifolds, the intersection form $\int \frac{F}{2\pi} \wedge \frac{F}{2\pi}$ is even and so $\frac{1}{2}\int \frac{F}{2\pi} \wedge \frac{F}{2\pi}$ is an integer.

so $\quad e^{i(\theta+2\pi)\,[\text{instanton no.}]} = e^{i\theta\,[\text{instanton no.}]}$

T-TRANSFORMATION $\longrightarrow$ shifts $\theta \to \theta + 2\pi$

$$F \longmapsto F$$

$$(n,m) \longmapsto (n-m, m)$$

shift is due to the Witten effect we saw before

$$\tau \longmapsto \tau + 1$$

$$\begin{pmatrix} -\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right) \\ F \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix}$$

Integrating $\longrightarrow \begin{pmatrix} -n \\ m \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -n+m \\ m \end{pmatrix}$

• The S- and T- transformations together generate the group $SL(2,\mathbb{Z})$

S, T generate $SL(2,\mathbb{Z})$ electric-magnetic duality.

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$M \in SL(2,\mathbb{Z}) \qquad \alpha\delta - \beta\gamma = 1$$

$$\alpha, \beta, \gamma, \delta \in \mathbb{Z}$$

The field strength F and the dual field strength transform as $SL(2,\mathbb{Z})$ doublets

$$\begin{pmatrix} -\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right) \\ F \end{pmatrix} \longmapsto M \begin{pmatrix} -\left(\frac{4\pi}{e^2} * F - \frac{\theta}{2\pi} F\right) \\ F \end{pmatrix}$$

$$\begin{pmatrix} -n \\ m \end{pmatrix} \longmapsto M \begin{pmatrix} -n \\ m \end{pmatrix}$$

$$(m, n) \longmapsto (m, n) M^{-1}$$

$$\tau \longmapsto M\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

Fractional Linear Transformations.

▸ The combination $(n_1 m_2 - n_2 m_1)$ is invariant under $SL(2,\mathbb{Z})$ transformations.

▸ $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad S^2 = -\mathbb{1}$

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad (ST)^3 = -\mathbb{1}$

▸ $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $\qquad \tau \mapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$. If $(\alpha, \beta, \gamma, \delta) \to (-\alpha, -\beta, -\gamma, -\delta)$ then the transformation of τ does not change.

$\Rightarrow$ Only the subgroup $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z}) / \{\mathbb{1}, -\mathbb{1}\}$ acts on τ.

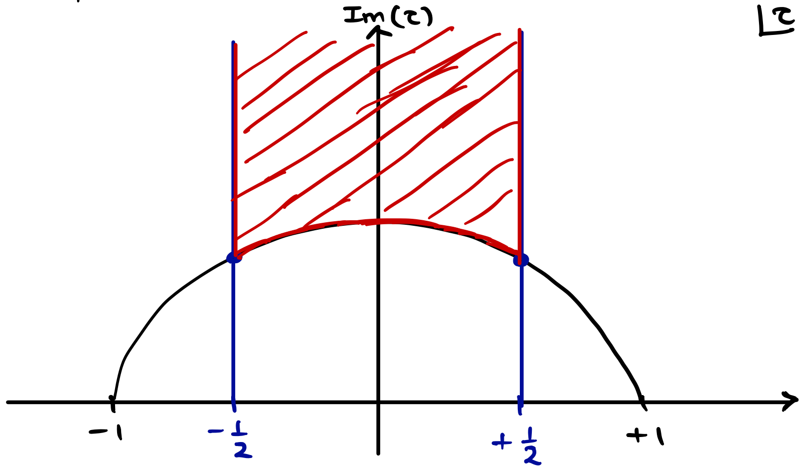
▸ The electric-magnetic duality transformation should be a viewed as a change of frame, and is <u>not</u> a symmetry. It yields an equivalent description of the theory. It is particularly useful as the S transformation is actually a strong-weak coupling transformation.

$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}$ so $\text{Im}(\tau) > 0$

Using $PSL(2,\mathbb{Z})$ on τ, we can always find a "DUALITY FRAME" where the complexified coupling τ belongs to the Fundamental Domain $H^+/PSL(2,\mathbb{Z})$.

Here $H^+ = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ (upper half plane)

- So, what is this fundamental domain?



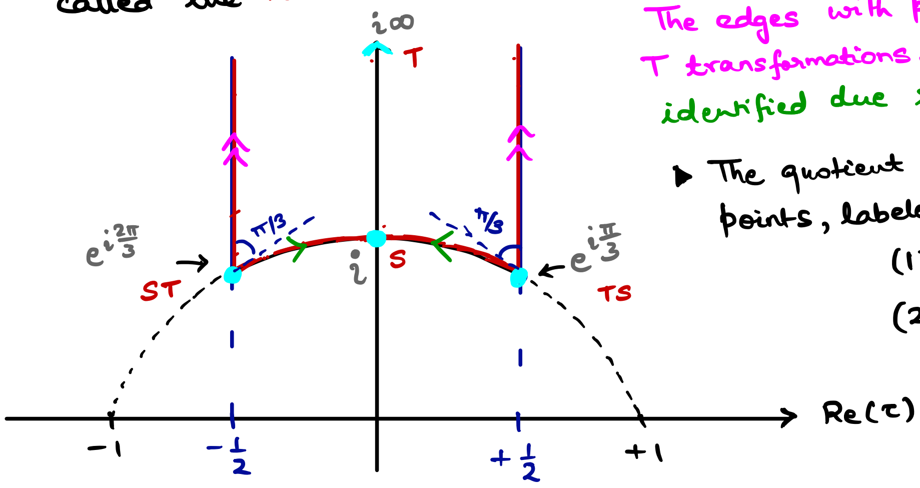
Recall that $T: \tau \to \tau + 1$

Using the T-transformation, we can always make $\text{Re}(\tau)$ lie in the range $[-\frac{1}{2}, +\frac{1}{2}]$ or equivalently make θ lie in the range $[-\pi, \pi]$.

But also $S: \tau \to -\frac{1}{\tau}$

so if $|\tau| > 1$ then S maps it to a τ' that lies inside the half-circle shown.

- The key idea is that by using S and T transformations on τ (or more precisely $PSL(2,\mathbb{Z})$ transformations), we can always find a description in which τ lies in the shaded red region of the diagram above. This red region is called the Fundamental Domain.



The edges with pink arrows are identified due to T transformations. The arcs with green arrows are identified due to S transformations.

- The quotient $H^+/PSL(2,\mathbb{Z})$ has 3 fixed points, labeled in the figure.

(1) $\tau = i$ fixed by S. opening angle $= \frac{2\pi}{2} = \pi$

(2) $\tau = e^{i\frac{\pi}{3}}$ fixed by TS

$TS: \tau \to \frac{\tau - 1}{\tau}$

(so $\tau^2 - \tau + 1 = 0$ for fixed pt.

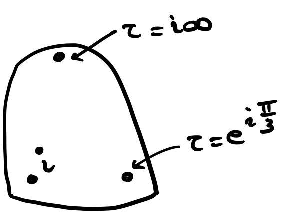
$\Rightarrow \tau = \frac{1 \pm i\sqrt{3}}{2} = e^{\pm i\pi/3}$

opening angle $= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

(3) $\tau = e^{i\frac{2\pi}{3}}$ fixed by ST (equivalent to (2)).

(4) $\tau = i\infty$ fixed by T. opening angle $= \frac{2\pi}{\infty} = 0$

Topologically, the fundamental domain can be represented as a sphere with 3 special points $(i, e^{i\frac{\pi}{3}}, i\infty)$



There are 3 orbifold singularities as we started from a smooth space and modded out by a discrete transformation. At these singularities, there is a conical deficit angle.

The opening angle is related to the order of the transformation in the group which leaves the object fixed.

e.g. (1) $\tau = i$ is fixed by S. The order of S is 2 in $PSL(2,\mathbb{Z})$ (as $S^2=-1$) so the deficit angle is $\frac{2\pi}{2} = \pi$.

(2) $\tau = e^{i\frac{\pi}{3}}$ is fixed by TS. The order of TS is 3 so the deficit angle is $\frac{2\pi}{3}$.

(3) $\tau = i\infty$ is fixed by T. But the order of T is ∞ so the deficit angle is $\frac{2\pi}{\infty} = 0$.

The deficit angle at the cusps equals $2\pi/$(order of the little group or stabilizer group that leaves the cusps fixed).

- We will now look at non-abelian gauge theories that can be spontaneously broken to abelian gauge theories at low energies.
- In abelian gauge theory, there is no way to introduce a magnetically charged source that couples to the theory. One can only introduce an electrically charged source. The Dirac monopoles then are point-like objects with infinite energies. They do not arise as dynamical states in a theory of electromagnetism.
- 't-Hooft and Polyakov in the 70s discovered that in case non-abelian gauge theories can be spontaneously broken to abelian gauge theories, one can have magnetic monopoles which are completely smooth and are finite energy solutions. They are actual excitations of the theory.

BOSONIC GEORGI-GLASHOW MODEL (for now, $\theta = 0$)

$SU(2)$ gauge theory, Φ in adjoint representation (TRIPLET)
$\simeq$
$SO(3)$

this trace is in fundamental of $SU(2)$

$$\mathcal{L} = \frac{1}{g^2}\left(-\frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} + \operatorname{tr} D_\mu\phi D^\mu\phi - V(\Phi)\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu + i[A_\mu, \cdot\,]$$

$$\boxed{V(\Phi) = \underbrace{\lambda}_{\lambda > 0}\left(\frac{1}{2}\operatorname{Tr}\Phi^2 - a^2\right)^2}$$

very similar to the Higgs potential except now Φ is an $SU(2)$ triplet rather than a doublet.

V is minimized by
($V=0$)

$$\Phi = a\sigma_3 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$ up to gauge transformations.

$\hookrightarrow$ breaks $SU(2) \longrightarrow U(1)$ Adjoint Higgs Mechanism.

In the standard Higgs mechanism, the doublet breaks $SU(2)$ completely.
But in this case, as Φ is in the adjoint representation, it transforms as

$$\Phi \mapsto U \Phi U^{-1}$$

so this $U(1)$ along the σ_3 direction yields the above expectation value.

Set

$$A_\mu = \begin{pmatrix} A_\mu^{U(1)} & W_\mu^+ \\ W_\mu^- & -A_\mu^{U(1)} \end{pmatrix}$$

diagonal parts give the gauge field for the low energy massless $U(1)$

$$= A_\mu^{U(1)} \sigma_3 + W_\mu^+ \sigma_+ + W_\mu^- \sigma_-$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Similarly, we can expand Φ as

$$\Phi = a\sigma_3 + (\underbrace{H}_{\text{Higgs field}}\sigma_3 + \delta\phi^+ \sigma_+ + \delta\phi^- \sigma_-)$$

$\delta\phi^+$, $\delta\phi^-$: eaten by $W_\mu^\pm$, making them massive

W_μ^+, W_μ^- have $U(1)$ charge as measured by the coupling to $A_\mu^{U(1)}$ $= \pm 2$.

$\Rightarrow W_\mu^+, W_\mu^-$ have $U(1)$ electric charges $= \pm 2$.

[Usually, $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ $q_1 \leftarrow$ electric charge 1, $q_2 \leftarrow$ electric charge -1. In our theory, there are no doublets.]

Perturbative Spectrum

	Mass	Spin	$U(1)$ charge
H	$2a\sqrt{\lambda}$	0	0
$A_\mu^{U(1)}$	0	± 1	0
$W_\mu^\pm$	$2a$	1	± 2

2: absolute value of charge; a: v.e.v of Higgs field

't-Hooft and Polyakov monopoles

The above non-abelian gauge theory has some static (time-independent; and $A_t = 0$ gauge choice), finite energy solutions of the equations of motion, called Solitons. They carry magnetic charges under the low-energy massless $U(1)$ theory.

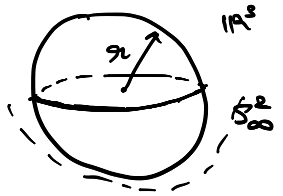
- **Finite Energy Solutions** $\Rightarrow \int d^3r\, T_{00} < \infty$. $\therefore T_{00} \xrightarrow[r \to \infty]{} 0$ faster than r^{-3}.

Therefore at $r \to \infty$, $F_{\mu\nu} \to 0$, $D_\mu \phi \to 0$, $V(\phi) \longrightarrow \underset{(\text{minimum})}{0}$

$$\Rightarrow \phi \to U \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} U^{-1} \qquad U \to \text{gauge transformation}$$

LECTURE III

't-Hooft Polyakov Monopoles



Static, finite energy solutions of e.o.m.

Finite Energy $\quad T_{00} \xrightarrow[r\to\infty]{} 0$ fast enough

$r \to \infty$, $\quad F_{\mu\nu} \to 0$, $\quad D_\mu \phi \longrightarrow 0$, $\quad V(\phi) \longrightarrow 0$, $\quad \Phi \xrightarrow[r\to\infty]{} \Phi_\infty$

The solution is supported by a nontrivial topology of Φ.

$$\Phi_\infty : \underset{\text{SPATIAL 2-SPHERE}}{S^2_\infty} \longrightarrow M_0 = \left\{ \Phi \,\middle|\, \tfrac{1}{2}\mathrm{tr}\,\Phi^2 = a^2 \right\} \simeq \underset{\text{FIELD SPACE}}{S^2}$$

This map is described by a "Winding number" (degree) of the map.

$$\pi_2(S^2) = \mathbb{Z}$$

e.g. $\Phi_\infty = a\,\hat{r}\cdot\vec{\sigma}$ has winding number $= 1$.
has winding number in field space.

It goes once around the two-sphere in field space.

We can write this as

$$\Phi_\infty = e^{-iT}\underbrace{(a\sigma_3)}_{\text{field configuration of vacuum}} e^{iT} \qquad \text{where } T = \frac{\theta}{2}\left(-\sigma^1 \sin\varphi + \sigma^2 \cos\varphi\right)$$

e^{-iT} : gauge transformation

singular at $\theta = 0$ & $\theta = \pi$ (spherical coordinates do not define φ unambiguously at the poles)

$$\begin{pmatrix} \text{WINDING NUMBER} \\ \text{OF } \Phi_\infty \end{pmatrix} = \begin{pmatrix} \text{Magnetic charge} \\ \text{for } \boxed{F^{U(1)}_{\mu\nu} = \frac{1}{2a}\mathrm{tr}\,(\Phi F_{\mu\nu})} \end{pmatrix}$$

$$F^{U(1)}_{\mu\nu} = \frac{1}{2a}\mathrm{tr}\,(\Phi F_{\mu\nu})$$

This unbroken U(1) varies along S^2_∞.

For a particular direction $\hat{r}$, parametrized by θ, φ, the map $\Phi_\infty = a\,\hat{r}\cdot\vec{\sigma}$ fixes a direction in field space; and $F^{U(1)}_{\mu\nu}$ determines the U(1) gauge field for this choice.

at $r \to \infty$, $\quad d\Phi_\infty \neq 0$ along S^2_∞.

Recall that for finite energy, we required $D_\mu \Phi_\infty \longrightarrow 0$ as $r \to \infty$.

$$0 = D\Phi_\infty = d\Phi_\infty - i\left[A_\infty, \Phi_\infty\right] \Rightarrow \text{need a nontrivial } A_\infty \neq 0 \text{ along } S^2_\infty.$$

In fact as $r \to \infty$, $\quad A^{U(1)}$ looks like a Dirac Monopole. So, it carries a magnetic charge.

- As $r \to 0$, the field configuration is regular thanks to non-abelian degrees of freedom. [This is in contrast to the Dirac monopole for a U(1) gauge theory in isolation.]

So the fact that the U(1) theory can be embedded inside a bigger non-abelian gauge theory allows the possibility of having 't-Hooft Polyakov monopole solutions, which are smooth field configurations.

't-Hooft Polyakov ansatz

At ∞, the scalar field looks like $\Phi_\infty = a\,\hat{r}\cdot\vec{\sigma}$, and the gauge field is such that its component along the unbroken U(1) is a Dirac Monopole.

Ansatz :

$$\phi(\vec{r}) = \frac{\vec{r}\cdot\vec{\sigma}}{r^2} H(ar)$$

$$A(\vec{r}) = \epsilon_{ijk}\frac{\sigma^i x^j dx^k}{r^2}\left(1 - K(ar)\right)$$

Look SU(2) space rotations (indices of x^j, dx^k) w/ SU(2) gauge rotations (indices of σ)

Now we will impose boundary conditions to ensure regularity and finite energy. Let $\xi = ar$.

$\xi \to \infty$

$$\Phi \to \Phi_\infty \qquad A \to A^{U(1)}_{\text{Dirac Monopole}} \quad \Rightarrow \quad \begin{array}{l} H(\xi) - \xi \to 0 \\ K(\xi) \to 0 \end{array}$$

decay to zero is up to exponentially suppressed corrections

$(e^{-M_W r}$ for A & $e^{-M_H r}$ for $\phi)$

$\xi \to 0$

$$H(\xi) = O(\xi)$$
$$K(\xi) - 1 = O(\xi)$$

Solving the full equations of motion is hard, but using these boundary conditions, one can make more progress using the above ansatz. When we do this, we find that for small radius (small ξ),

$$\left.\begin{array}{l} H(\xi) \sim \xi^2 \\ K(\xi) - 1 \sim \xi^2 \end{array}\right| \text{ for small } \xi$$

This ensures regularity. Also,

$$\left.\begin{array}{l} \Phi(0) \to 0 \\ A(0) \to 0 \end{array}\right| \Rightarrow$$ there is the full unbroken SU(2) at the core of the monopole

---

- Now, let us study a different ansatz : Φ = as above
  Julia-Zee ansatz $\quad A = (\text{as above}) + \frac{\vec{r}\cdot\vec{\sigma}}{r^2} J(ar)\,dt$

▸ Repeating the above analysis, we will find that the ansatz carries both electric as well as magnetic charges. This field configuration therefore describes DYONS.

▸ Using regularity and finite energy, we can put constraints on the asymptotics of J.

- At the classical level, the electric charge is not quantized but at the semiclassical level, it is indeed quantized.

Bogomoln'yi – Prasad – Sommerfield Bound (BPS)

We want to compute the energy of the field configuration.

$$g^2 \times (\text{Energy}) = \int d^3x \, \mathrm{tr}\left(E_i^2 + B_i^2 + (D_i\phi)^2 + (D_0\phi)^2 + V(\phi)\right]$$

$\parallel$ Mass

$$= \int d^3x \, \mathrm{tr}\Big[\underbrace{(E_i - \sin\theta \, D_i\phi)^2}_{\geq 0} + \underbrace{(E_i - \cos\theta \, D_i\phi)^2}_{\geq 0} + \underbrace{(D_0\phi)^2}_{\geq 0}\Big]$$

$$+ \int d^3x \, V(\phi) + 2\int d^3x \, \mathrm{tr}\left(\sin\theta \, E_i D_i\phi + \cos\theta \, E_i D_i \phi\right)$$

can we pull D_i out!

< θ is just some auxiliary angular variable that we've introduced at this stage. >

EoM $\Rightarrow$ $D_i E_i = 0$, $D_i B_i = 0$ (follow from eqns of motion)

Sloppy? conservation conditions

$$g^2 \times (\text{Energy}) \geq 2\int d^3x \left[\sin\theta \, \vec{\nabla}\cdot \mathrm{tr}(\vec{E}\phi) + \cos\theta \, \vec{\nabla}\cdot\mathrm{tr}(\vec{B}\phi)\right]$$

The right-hand side is now expressed in terms of the electric and magnetic charges of the unbroken U(1).

magnetic field of unbroken U(1) — magnetic field of the whole SU(2) group

$$2\pi m = \oint_{S^2} \vec{B}^{U(1)} \cdot d\vec{s} = \frac{1}{2a}\oint \mathrm{tr}(\phi\vec{B})\cdot d\vec{s}$$

electric field of the whole SU(2) group

$$2\pi n = \oint_{S^2} \frac{4\pi}{e^2}\vec{E}^{U(1)} \cdot d\vec{s} = \frac{1}{2a}\frac{8\pi}{g^2}\oint \mathrm{tr}(\phi\vec{E})\cdot d\vec{s}$$

electricfield of unbroken U(1)

There is a small subtlety in the second equation, regarding the coupling used.

$$\frac{1}{g^2} F_{\mu\nu}^{U(1)\,2} = \frac{1}{2e^2} F_{\mu\nu}^{U(1)^2}$$

coupling for SU(2) — coupling for U(1) — chosen to make n an integer

Using these charges, we find that

$$(\text{Energy}) \geq a\left(\sin\theta \cdot n + \cos\theta \cdot \frac{4\pi}{e^2} m\right) \quad \forall\,\theta$$

$$\xrightarrow[\text{RHS}]{\text{max}} \boxed{(\text{energy}) \geq a\sqrt{n^2 + \left(\frac{4\pi}{e^2}\right)^2 m^2}}$$

$$\sin\theta = \frac{n}{\sqrt{n^2 + \left(\frac{4\pi}{e^2}\right)^2 m^2}}$$

$$\cos\theta = \frac{(4\pi/e^2)m}{\sqrt{\cdots}}$$

BPS Bound

depends on
1. vev of the scalar field
2. electric charge
3. magnetic charge

- BPS bound is saturated when

$\lambda \to 0$ (BPS limit) fixed a $\to$ makes potential vanish

$D_0 \phi = 0$ $\qquad E_i = \sin\theta\, D_i\phi$ $\qquad \tan\theta = \dfrac{n}{\frac{4\pi}{e^2} m}$ $\qquad \boxed{\text{BPS equations}}$

$B_i = \cos\theta\, D_i \phi$

If you saturate the BPS bound, you get equations of first order, which imply the equations of motion and minimize the energy. In contrast, the original equations of motion were second order. + extremize the action in a fixed topological sector

Now let us turn on the Θ-angle, i.e. $\Theta \neq 0$ [By Θ here, we're referring to the Θ angle term in the action, not the θ angle introduced above.]

$$\sqrt{n^2 + \left(\frac{4\pi}{e^2} m\right)^2} \xrightarrow{\text{replace}} |n + m\tau^{U(1)}|^2$$

$$\text{where} \quad \tau^{U(1)} = \frac{\Theta^{U(1)}}{2\pi} + i\frac{4\pi}{e^2_{U(1)}}$$

Comparing the complexified coupling of the U(1) to the complexified coupling of the whole SU(2) group, one finds

$$\tau^{U(1)} = 2\tau^{SU(2)}$$

- $\boxed{\text{Mass} \geq a\,|n + m\tau^{U(1)}|}$ saturated for the solution of the BPS equations

- BPS bound and equations appear with $\mathcal{N} = 2$ SUSY.

PART 2: BASICS OF $\mathcal{N}=2$ SUSY

4d $\mathcal{N}=2$ SUSY ALGEBRA

$$\boxed{\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot\alpha J}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^I_J \\ \{Q^I_\alpha, Q^J_\beta\} &= 2\varepsilon_{\alpha\beta} \bar{Z}^{IJ} \\ &= 2\varepsilon_{\alpha\beta}\varepsilon^{IJ}\bar{Z} \\ \{\bar{Q}_{\dot\alpha I}, \bar{Q}_{\dot\beta J}\} &= 2\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{IJ} Z \end{aligned}}$$

Q^I_α $\quad I = 1, 2 = \mathcal{N}$

$\bar{Q}_{\dot\alpha I}$ $\quad \alpha, \dot\alpha$ $\quad SL(2)\times SL(2)$

$\sigma^\mu = (\mathbb{1}, \vec{\sigma})$

mostly minus metric

*since $I, J = 1, 2$ we can replace $\bar{Z}^{IJ}$ with $\varepsilon^{IJ} \times$ a number

- Z is the "central charge". It is called "central" because it commutes with all generators of the superalgebra, so it belongs to the center of the superalgebra.

So, for any irreducible representation, it can be diagonalized.

R-Symmetry : automorphism of the SUSY algebra (some transformation that leaves the SUSY algebra invariant)

$\hookrightarrow SU(2)_R \times U(1)_R$ $\quad$ ($= U(2)_R$ because we can rotate Q^1_α into Q^2_α, etc.) and $U(2) = SU(2)\times U(1)$

$Q^I : \left(\frac{1}{2}\right)_{-1}$ $\leftarrow$ $U(1)$ R-charge

$Z : (0)_2$ $\leftarrow$ Z is a singlet under $SU(2)$

Within each multiplet, $Z \in \mathbb{R}_{>0}$

Massless Irreps

$p^\mu = (E, 0, 0, E)$

$$\{Q^I_\alpha, \bar{Q}_{\dot\alpha J}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\alpha} \delta^I{}_J$$

$\alpha, \dot\alpha = 1$

$$\langle \text{phys} | \{Q^I_1, \bar{Q}_{\dot{1} I}\} | \text{phys} \rangle = 0$$

because $\{Q^I_1, \bar{Q}_{\dot 1 J}\} = 0$ by SUSY algebra

$$\Rightarrow \| Q^I_1 |\psi\rangle\|^2 + \|(Q^I_1)^\dagger |\psi\rangle\|^2 = 0 \quad \text{for all physical states } |\psi\rangle$$

$$\Rightarrow Q^I_1 |\psi\rangle = 0 \quad \text{and} \quad (Q_1^I)^\dagger |\psi\rangle = 0$$

$$\Rightarrow \langle \psi | Z | \psi \rangle = 0$$

$\rightarrow$ we'll set $Z = 0$ from now on.

Define

$$a^I = \frac{1}{\sqrt{2E}} Q_2^I \quad , \quad (a^I)^\dagger = \frac{1}{\sqrt{2E}} \bar{Q}_{\dot{2}I}$$

a^I ↓ lowers helicity, $U(1)_R$ charge

$(a^I)^\dagger$ ↓ raises helicity, $U(1)$ R-charge

$$\{ a^I , (a^J)^\dagger \} = \delta^{IJ}$$

others $= 0$

As $Q, \bar{Q}$ are fermionic, so are a & $a^\dagger$

So we have 2 fermionic raising and lowering operators

To build a representation, we start from a vacuum $|\lambda\rangle$ s.t. $a^I |\lambda\rangle = 0$

$|\lambda\rangle$: ↑ helicity, → $U(1)_R$ charge r

two of these → $(a^I)^\dagger |\lambda\rangle = |\lambda + \frac{1}{2}\rangle$ $\quad r+1$

one of this → $\frac{1}{2} \varepsilon_{IJ} (a^I)^\dagger (a^J)^\dagger |\lambda\rangle = |\lambda + 1\rangle$ $\quad r+2$

$|\lambda| \leq 1$ → avoids gravitons & gravitinos

$\lambda = 0$

$$\left(|0\rangle \oplus 2 \left|\tfrac{1}{2}\right\rangle \oplus |1\rangle \right) \oplus \underbrace{\left(|-1\rangle \oplus 2\left|-\tfrac{1}{2}\right\rangle \oplus |0\rangle \right)}_{\text{CPT conjugate}}$$

⇒ • 2 real scalars (or 1 complex scalar)
• 2 Weyl fermions
• 1 massless gauge boson

} $\mathcal{N}=2$ massless vector multiplet = ($\mathcal{N}=1$ vector) ⊕ ($\mathcal{N}=1$ chiral)

$\lambda = -1/2$

self CPT conj. (half-hypermultiplet) → pseudoreal rep?

$$\left(\left|-\tfrac{1}{2}\right\rangle \oplus 2|0\rangle \oplus \left|\tfrac{1}{2}\right\rangle \right) \oplus \underbrace{\left(\left|-\tfrac{1}{2}\right\rangle \oplus 2|0\rangle \oplus \left|\tfrac{1}{2}\right\rangle \right)}_{\text{CPT conjugate ?}}$$

⇒ • 4 real scalars (or 2 complex scalars)
• 2 Weyl fermions

} $\mathcal{N}=2$ massless hypermultiplet = ($\mathcal{N}=1$ chiral) ⊕ ($\mathcal{N}=1$ chiral in conj. rep.)

1:08:45

Massive Irreps.

$P^\mu = (M,0,0,0)$

- breaks $SL(2)\times SL(2) \longrightarrow SL(2)$
- identifies dotted with undotted : $\alpha = \dot{\alpha}$

$$\boxed{\begin{aligned}\{Q^I_\alpha, (Q^J_\beta)^+\} &= 2M\,\delta_{\alpha\beta}\,\delta^{IJ} \\ \{Q^I_\alpha, Q^J_\beta\} &= 2\,\varepsilon_{\alpha\beta}\,\varepsilon^{IJ} Z \\ \{(Q^I_\alpha)^+, (Q^J_\beta)^+\} &= 2\,\varepsilon_{\alpha\beta}\,\varepsilon_{IJ} Z\end{aligned}}$$

[this should really be $\delta^I{}_J$ I think]

Rotate Z to be real and >0

Let us rewrite the SUSY algebra in a different form. Define

$$a_\alpha = \frac{1}{2}\left(Q^1_\alpha + \varepsilon_{\alpha\beta}(Q^2_\beta)^+\right) \qquad b_\alpha = \frac{1}{2}\left(Q^1_\alpha - \varepsilon_{\alpha\beta}(Q^2_\beta)^+\right)$$

$$a^+_\alpha = \dots \qquad b^+_\alpha = \dots$$

We find

$$\boxed{\begin{aligned}\{a_\alpha, a^+_\beta\} &= \delta_{\alpha\beta}(M+Z) \\ \{b_\alpha, b^+_\beta\} &= \delta_{\alpha\beta}(M-Z) \\ \text{others} &= 0\end{aligned}}$$

Now if we set $\alpha=\beta$ in say, the second anticommutation relation, we get

$$M \geqslant Z$$

whereas doing the same in the first relation yields

$$M \geqslant -Z$$

so,

$$\boxed{M \geqslant |Z|}$$

BPS Bound

If Z were not rotated to be real and positive, it could've been negative.

Notice the BPS bound is a consequence of the supersymmetry algebra.

There are two cases to consider:

① $M > |Z|$ Long Multiplet (2 raising + 2 lowering operators)

acting with a^+_α, b^+_α

16 states (8 bosonic + 8 fermionic) $\rightarrow$ Long massive vector multiplet

② $M = |Z|$ Short Multiplet (1 raising + 1 lowering operator)

4 states (2 bosonic + 2 fermionic) + conj.

same # d.o.f. as massless multiplets

two possibilities
- massive hypermultiplet
- short massive vector multiplet

NOTE : Short MASSIVE multiplets have as many d.o.f. as MASSLESS multiplets.

So, starting from a massless hyper or vector, one can give them a mass while at the same time, keeping the multiplet short.

→ Massless multiplets can gain mass and remain short (BPS).

From now on, we will be dealing mostly with short multiplets, either massless multiplets or massive hypermultiplets, or short massive vector multiplets.

PROPERTIES OF BPS (SATURATED) STATES

$$\boxed{M = |Z|}$$

① The BPS relation is not affected by continuous deformations, e.g. changing the coupling, moduli, $\hbar$

↳ in changing these things, Z does indeed change but to begin with, one has 4+4 states (as a BPS multiplet) if $M \neq |Z|$ now due to the change, it is no longer BPS and hence is a long multiplet ; but long multiplets have more states which cannot come out of nowhere.

Short multiplets remain short. But there is a small caveat

② DECAY OF BPS States?

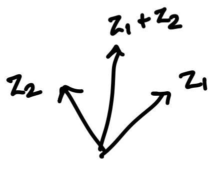
Consider a BPS state with central charge Z and mass $M = |Z|$. Go to the Rest frame

central charge conservation

$$Z = Z_1 + Z_2$$

$$M = |Z| = |Z_1 + Z_2| \leq |Z_1| + |Z_2| \leq M_1 + M_2$$

Triangle Inequality } w/out assuming that the final states are BPS (if they were, we'd have $M_1 = |Z_1|$, $M_2 = |Z_2|$ for them)



$$\Rightarrow M \leq M_1 + M_2$$

So the decay can happen if and only if

① $M = M_1 + M_2$

and at the same time

② $M_1 = |Z_1|$, $M_2 = |Z_2|$

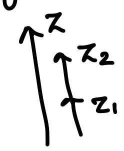
i.e. the decay products are BPS.

③ all inequalities are saturated

Decay can happen $\Longleftrightarrow$

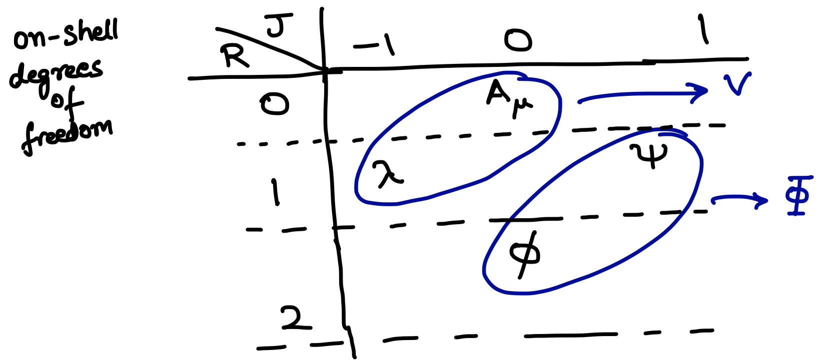
$$\begin{array}{ccccc} M & = & M_1 & + & M_2 \\ \| & & \| & & \| \\ |Z| & & |Z_1| & & |Z_2| \end{array}$$

and Z, Z_1, Z_2 are aligned



MASSLESS VECTOR MULTIPLET $\Psi = (V, \Phi)$

V: $N=1$ vector multiplet; Φ: $N=1$ chiral multiplet in adj. rep^n.



$U(1)_J \subset SU(2)_R$

$J = 2J_3$

$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$

$$\mathcal{L}_{YM} = \frac{1}{4\pi} \mathrm{Im}\left[\tau \,\mathrm{Tr}\left(\int d^2\theta\, W^\alpha W_\alpha + \underbrace{\int d^4\theta\, \Phi^+ e^V \Phi}_{\text{kinetic term for scalar}}\right)\right]$$

W^α is the $\mathcal{N}=2$ superfield

in components

$$= \mathrm{Tr}\Big(\underbrace{-\frac{1}{2g} F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell YM term}} \underbrace{- \frac{\theta}{16\pi^2} F_{\mu\nu}(*F)^{\mu\nu}}_{\theta\text{-angle term}} + \underbrace{\frac{1}{g^2}(D_\mu\phi)^*(D^\mu\phi)}_{\text{kinetic term for scalars}}$$
$$+ \frac{1}{g^2}D^2 + \frac{1}{g^2} D[\phi, \phi^+] + \frac{1}{g^2}F^*F\Big)$$

+ fermions

(quite possibly upto factors of 2)

The Tr here is in the fundamental representation.

Here we have used $\mathcal{N}=1$ superspace. But in fact for the $\mathcal{N}=2$ vector multiplet, there is an $\mathcal{N}=2$ superspace.

$\mathcal{N}=2$ superspace $\quad \theta, \bar\theta, \quad \tilde\theta, \bar{\tilde\theta}$

$\mathcal{N}=2$ vector multiplet $\quad$ "$\mathcal{N}=2$ chiral" $\quad \bar D_\alpha \Psi = 0$, $\quad \tilde{\bar D}_\alpha \Psi = 0$

$$\Psi = \Phi(\tilde y, \tilde\theta) + \sqrt{2}\,\tilde\theta^\alpha W_\alpha(\tilde y, \tilde\theta) + \tilde\theta^2 \Psi_F(\tilde y, \tilde\theta)$$

$$\tilde y^\mu = x^\mu + i\tilde\theta\sigma^\mu\bar{\tilde\theta} + i\bar\theta\sigma^\mu\theta \quad ??$$

$$\Psi_F(\tilde y, \theta) = \Phi^+(\tilde y - i\theta\sigma\bar\theta, \bar\theta)\, e^V(\tilde y - i\theta\sigma\bar\theta, \bar\theta)\Big|_{\bar\theta\bar\theta}$$

auxiliary field

$$\boxed{\mathcal{L}_{YM} = \frac{1}{4\pi}\mathrm{Im}\int d^2\theta\, d^2\tilde\theta\; \frac{1}{2}\tau\Psi^2}$$

quadratic for renormalization

$\leftarrow$ manifestly $\mathcal{N}=2$ supersymmetric

▸ So just like we could write, for $\mathcal{N}=1$ chiral superfields the manifestly $\mathcal{N}=1$ supersymmetric integral

$$\int d^2\theta \; W(\Phi)$$

← holomorphic fxn of $\mathcal{N}=1$ chiral superfield

we can write the integral

$$\frac{1}{4\pi}\int d^2\theta \, d^2\tilde{\theta} \; \mathcal{F}(\Phi)$$

← holomorphic fxn of $\mathcal{N}=2$ chiral superfield ($\mathcal{N}=2$ vector multiplet?)

for $\mathcal{N}=2$ chiral superfields, which is manifestly $\mathcal{N}=2$ supersymmetric.

▸ So, the most general Lagrangian for the $\mathcal{N}=2$ vector multiplet is of the form

$$\boxed{\mathcal{L}_{YM} = \frac{1}{4\pi}\int d^2\theta \, d^2\tilde{\theta} \; \underline{\underline{\mathcal{F}(\Psi)}}}$$

→ determined by a single holomorphic fxn

PREPOTENTIAL (holomorphic)

In the classical case, the prepotential is just $\frac{1}{2}\tau\Psi^2$ as discussed above.

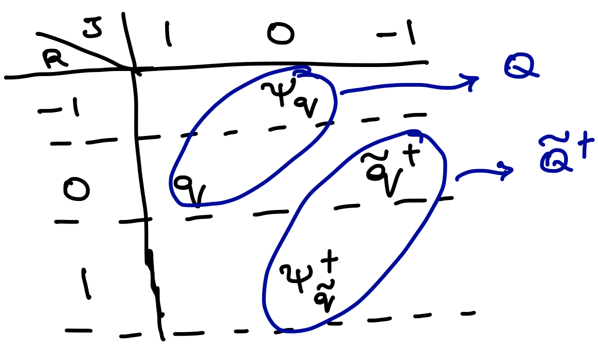
- The $U(1)_R$ charge of the prepotential has to be 4, $R[\mathcal{F}]=4$
- $\mathcal{F}$ is an $SU(2)_R$ singlet. Not surprising as Φ was an $SU(2)_R$ singlet.

<u>HYPERMULTIPLET</u> $\quad H = (Q, \tilde{Q}^{+})$

two chiral

Q & $\tilde{Q}$ are not supercharges. You can think of them as quarks

on-shell degrees of freedom:



- Q and $\tilde{Q}$ transform in conjugate representations.
- Q and $\tilde{Q}^{+}$ transform in the same representation

Pair of chirals Q, $\tilde{Q}$ in conj. representation of the gauge symmetry group and global symmetry group.

Several copies of hypermultiplets. Lagrangian for them in $\mathcal{N}=1$ notation

$$\boxed{\mathcal{L}_{hyper} = \int d^4\theta \Big(\underbrace{Q^{+i} e^{V} Q_i}_{\text{Kinetic term for } Q} + \underbrace{\tilde{Q}^{+i} e^{-V} \tilde{Q}_i}_{\text{Kinetic term for } \tilde{Q}} \Big) + \Big(\int d^2\theta \, \big(\tilde{Q}^{i} \, \Phi \, Q_i + m_i \tilde{Q}^{i} \tilde{Q}_i \big) + h.c. \Big)}$$

↑ $\mathcal{N}=1$ superpotential constrained by $\mathcal{N}=2$ SUSY

By $\mathcal{N}=2$ supersymmetry, the kinetic term and the superpotential term have to be related, essentially because V and Φ belong to the same $\mathcal{N}=2$ multiplet. This fixes the superpotential to be $\tilde{Q}^i \Phi Q_i + \cdots$

▶ $\mathcal{N}=2$ SUSY $\Rightarrow$ $W = \tilde{Q}^i{}_a \Phi^a{}_b Q^b{}_i + \underbrace{m_i{}^j}_{} \tilde{Q}^i{}_a Q^a{}_j$

gauge indices (a, b)

scalar in dynamical vector multiplet (gauge symmetry) dynamical

mass matrix

scalar in nondynamical vector multiplet (global symmetry external)

m & Φ have similar roles.

▶ $SU(2)_R$ symmetry $\Rightarrow$ $[m, m^\dagger] = 0 \longrightarrow$ diagonalize $m = \mathrm{diag}(m_i)$

Therefore one can write the superpotential mass term as $m_i \tilde{Q}^i Q_i$.

▶ effective complex mass of the hypermultiplet, which comes from the second derivative of the superpotential (contribution comes from the terms in W)

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$$m_{eff} \sim m + \langle \Phi \rangle = Z_{eff}$$

complex — m_{eff}

expectation value of Φ — $\langle \Phi \rangle$

effective central charge — Z_{eff}

effective superpotential mass $m = Z$ effective central charge

Note: There is an $\mathcal{N}=2$ superspace formulation for the hypermultiplet but it is quite complicated. It is not that useful for the following.

CLASSICAL
MODULI SPACE OF SUPERSYMMETRIC VACUA ($m_i = 0$)

↳ hypermultiplet = 0

Classical moduli space is the space of minimal scalar potential, and in particular since the scalar potential is the sum of squares of D-terms and F-terms, we have to set to zero the D-terms for each vector multiplet ($\mathcal{N}=1$ vector multiplet), and the F-terms

V vector multiplet
$$\begin{cases} D^a = 0 \rightarrow \frac{1}{g^2}[\phi, \phi^+] + QQ^+ - \tilde{Q}^+\tilde{Q} = 0 \\ F_\Phi = 0 \rightarrow Q\tilde{Q} = 0 \end{cases}$$

H hypermultiplet
$$\begin{cases} F_{\tilde{Q}} = 0 \rightarrow \Phi Q = 0 \\ F_Q = 0 \rightarrow \tilde{Q}\Phi = 0 \end{cases}$$

$\#$

classical moduli space $\Rightarrow \mathcal{M}_{class} = \{\text{solutions of } \#\}/G$ (G = gauge group)

$\equiv \mathcal{M}_{F,D}/G$ ← notation

Even the classical moduli space of vacuua can be very complicated and can contain different branches.

▶ **COULOMB BRANCH $\mathcal{M}_{coulomb}$** : obtained by setting the scalars in the hypermultiplet to zero. So, $Q = \tilde{Q} = 0$.

Therefore, the last 3 equations of $\#$ are trivially satisfied. The D-term equations tell us that the scalar ϕ in the vector multiplet can acquire a v.e.v but it has to commute with ϕ^+:

$$[\phi, \phi^+] = 0$$

→ Diagonalize $\Phi = \sum_{i=1}^{rk(G)} a_i H^i$ — H^i : generators of the Cartan subalgebra of the gauge Lie algebra.

"Coulomb" : $G \xrightarrow{\langle\Phi\rangle} U(1)^{rk(G)} \rtimes W_G$ ← Weyl group (acts on the scalars a_i) does not commute with $U(1)^{rk(G)}$

maximal torus → whose Lie algebra is the Cartan subalgebra

So we have a bunch of abelian massless vector multiplets, and the rest of the vector multiplets are massive. At low energies these U(1) vector multiplets will carry Coulomb force.

classical Coulomb branch

$$\mathcal{M}_{coulomb} \cong \mathbb{C}^{rk(G)}/W_G \cong \mathbb{C}^{rk(G)}$$

← extra discrete part of gauge group that survives and has to be modded out

parametrized by a_i's and there are as as many a_i's as the rank of G

Weyl group ↙ residual gauge transformations that we have to divide by.

Example $G = SU(N)$.

$\Phi = \text{diag}(\lambda_1, \ldots, \lambda_N)$; $\sum_{i=1}^{N} \lambda_i = 0$ (overparametrizing Φ)

Weyl Group $W_{SU(N)} = S_N$ permutes λ_i (part of the gauge symmetry)

(S_N = symmetric group of N elements)

So we can parametrize the Coulomb branch by the complex numbers λ_i, modulo the Weyl group. So in particular we can parametrize it by invariants of the Weyl group, i.e. the Casimir Invariants.

Casimir Invariants: $$U_K = \frac{1}{K} \text{Tr}(\phi^K) \qquad K = 2, \ldots, N$$

$$= \frac{1}{K} \sum_{i=1}^{N} \lambda_i^k \longrightarrow \text{clearly invariant under permutations}$$

give a parametrization of

$$\mathcal{M}_{\text{coulomb}} \cong \mathbb{C}^{\text{rk}(G)}/W_G \simeq \mathbb{C}^{\text{rk}(G)}$$

$K = 1 \Rightarrow U_1 = \text{Tr}(\phi) = 0$ because of $SU(N)$

(for $K > N$, one can use Cayley-Hamilton Theorem to reduce this object.)

▶ **HIGGS BRANCH $\mathcal{M}_{\text{Higgs}}$**: now the scalar in the vector multiplet is zero, i.e. $\phi = 0$, and the scalars Q and $\tilde{Q}$ in the hypermultiplet can acquire a v.e.v. and the F-term equation $Q\tilde{Q} = 0$ and the D-term equation $QQ^+ - \tilde{Q}^+\tilde{Q} = 0$.

In particular, we have to satisfy the

$$\boxed{\begin{gathered} \mu_{\mathbb{C}}(H, H^+) \equiv Q\tilde{Q} = 0 \\ H = (Q, \tilde{Q}^+) \\ \mu_{\mathbb{R}}(H, H^+) \equiv QQ^+ - \tilde{Q}^+\tilde{Q} = 0 \end{gathered}}$$

$\longrightarrow$ we wrote these as a complex eqn. that comes from the F-term and and a real eqn. that comes from the D-term

We can think of $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ as a triplet of $SU(2)_R$ symmetry: $\vec{\mu}$

Sometimes people call these triplets of the D-term equations of $N=2$ supersymmetry.

$$\boxed{\mathcal{M}_{\text{Higgs}} = \left\{ H = (Q, \tilde{Q}^+) \mid \vec{\mu}(H, H^+) = 0 \right\} / G}$$

$\rightarrow$ This construction is called the **HYPER KÄHLER QUOTIENT**

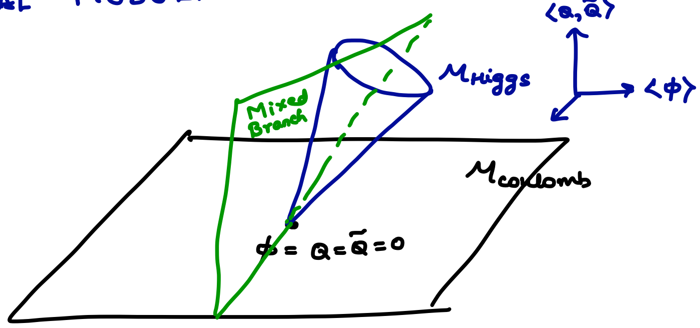
There is an interesting geometry behind the hyper Kähler quotient but it will not play an important role in what follows.

- For generic choice of Q and $\tilde{Q}$ satisfying these F & D-term equations, the gauge group G is completely Higgsed: there is no residual gauge group. This is why this is called the Higgs Branch.

▶ We can also have Mixed Branches, if $\langle \Phi \rangle \neq 0$
$\langle Q \rangle \neq \langle \tilde{Q} \rangle \neq 0$

In this case we will have nontrivial solutions to the F- and D-term equations for both Φ and $Q, \tilde{Q}$.

CLASSICAL MODULI SPACE $\mathcal{M}_{class}$, pictorially.....

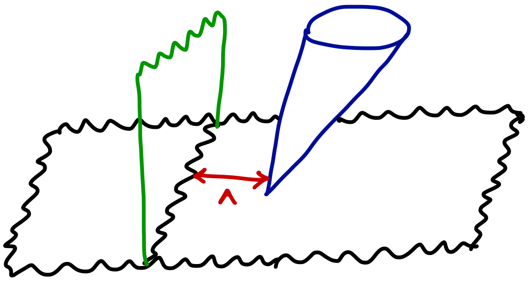


In the absence of masses, the equations for the Higgs branch are invariant under rescalings of $Q, \tilde{Q}$ ⇒ this is why the Higgs branch is visualized as a cone.

Locally, $\underbrace{\mathcal{M}_{class}}_{\text{classical moduli space}} = \underbrace{\mathcal{M}_V}_{\text{component in V where scalars in vector multiplet acquire v.e.v. and those in the hyper are frozen}} \times \underbrace{\mathcal{M}_H}_{\text{component in H where scalars in the hypermultiplet acquire v.e.v. and those in the vector multiplet are frozen}}$

QUANTUM MODULI SPACE OF SUPERSYMMETRIC VACUA

Let us first see what happens to the above picture



Λ: dynamically generated strong coupling scale of the theory.

Due to $N=2$ supersymmetry, there cannot be a superpotential. So, none of the original branches can be lifted. But they can be deformed due to quantum corrections.

Still, locally, $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$.

- Metric on $\mathcal{M}_V$ receives quantum corrections (one loop corrections + instanton corrections)
- Metric on $\mathcal{M}_H$ does not receive quantum corrections (so the dynamics on the Higgs branch isn't very interesting)
  → metric on $\mathcal{M}_H$ is classically exact!

$\mathcal{M}_V$: parametrized by scalars a^i in Abelian vector multiplets.

$\mathcal{M}_H$: parametrized by scalars $x^I, \tilde{x}_I$ in neutral hypermultiplet (if they were charged, they would Higgs the gauge group and would get eaten by W-bosons in the Higgs mechanism)

can be thought of as scalars in some gauge-invariant combination of the hypers

Using this information we can write down the structure of the low-energy effective action (for very low energies).

"INFRARED EFFECTIVE LAGRANGIAN"

$$\int d^4\theta \; K(X^I, \tilde{X}_I, X_I^+, \tilde{X}^{+I}) + \underbrace{\cdots}$$

K: Kähler potential for the superfields based on the scalars in the neutral hypermultiplet

$\underbrace{\cdots}$: discussed below (part that deals with the abelian vector multiplet)

[capital X denotes the chiral superfield that contains $x, \tilde{x}$] $\longrightarrow$ Metric on M_H is not affected by vector multiplets. (a^i, m_i, Λ)

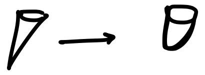
Λ: also sits in the vector multiplet

(if they were charged, we'd see something like $\int d^4\theta \; K(X, X^+ e^V)$)

so the $N=1$ vector multiplets V^i cannot appear as the scalars are neutral.

$\xRightarrow[\text{SUSY}]{N=2}$ $N=1$ chirals Φ^i also cannot appear here either (because $N=2$ SUSY relates V^i & Φ^i)

- We do not discuss Fayet-Iliopoulos terms. They only appear if the gauge group has a $U(1)$ factor but we do not like them for other reasons.
- FI terms can smoothen out the cone, as being dimensionful objects, they introduce a scale.



- FI terms will affect the metric on the Higgs branch or M_H, but there will still be no quantum corrections.

→ Now lets look at the part of the IR effective Lagrangian that depends on the abelian vector multiplet (the dots above)

Recall that the most general $N=2$ supersymmetric Lagrangian can be written in terms of a single holomorphic fxn, the PREPOTENTIAL.

$$\int d^4\theta \; K(X^I, \tilde{X}_I, X_I^+, \tilde{X}^{+I}) + \frac{1}{4\pi} \operatorname{Im}\left(\int d^2\theta \; \frac{1}{2} \mathcal{F}_{ij}(A) W^{i\alpha} W^j_\alpha + \int d^4\theta \; \bar{A}^i \mathcal{F}_i(A) \right)$$

gauge kinetic term for gauginos involving 2nd derivative of the prepotential

Kähler potential term for the scalars governed again by the prepotential

$A^i = a^i + \cdots$

$W^i_\alpha = \lambda^i{}_\alpha + \cdots$ gauginos

$$\mathcal{F}_i(A) = \frac{\partial \mathcal{F}}{\partial A^i}$$

$$\mathcal{F}_{ij}(A) = \frac{\partial^2 \mathcal{F}}{\partial A^i \partial A^j}$$

in terms of an IR effective prepotential $\mathcal{F}(A)$, which is a fxn. only of the massless abelian vector multiplets

- $\mathcal{F}(A)$ will be affected by one-loop perturbative corrections and nonperturbative instanton corrections.

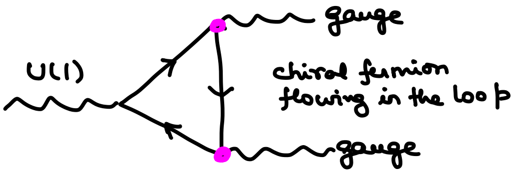
So it will be nontrivial to derive the form of the prepotential $\mathcal{F}(A)$.

What Seiberg & Witten did in 1994 was to determine this low-energy prepotential $\mathcal{F}(A)$ exactly.

We will need some information about the perturbative quantum corrections, in particular the anomaly of the $U(1)_R$-symmetry and the perturbative renormalization group of the gauge coupling.

• $U(1)_R$ ANOMALY

1-loop



$$\partial_\mu j^\mu_{U(1)_R} = \sum_{\substack{\text{Weyl}\\ \text{fermions}\\ \psi_i}} R[\psi_i] \cdot \frac{1}{16\pi^2} \, tr_{R_i} (F_{\mu\nu} * F^{\mu\nu})$$

R-charge of the Weyl fermions

representation of the gauge group that the fermions transform in

$$= \underbrace{\left(\sum_{\psi_i} R[\psi_i] \, T[r_i] \right)}_{\text{ANOMALY COEFFICENT 'A'}} \underbrace{\frac{1}{16\pi^2} tr_{fund} (F_{\mu\nu} * F^{\mu\nu})}_{\text{instanton number density}}$$

Recall some group theory

$$tr_{R_i} (T^a T^b) = C(R_i) \, \delta^{ab}$$

$$tr_{fund} (T^a T^b) = C(fund) \, \delta^{ab}$$

$$\Rightarrow \boxed{T(R_i) = \frac{C(R_i)}{C(fund)}} \longrightarrow$$ T-index is independent of anomaly (Dynkin index)

Example : Application of this formula to $\mathcal{N}=2$ gauge theory

Anomaly coefficient : $A = \sum_{\substack{\text{Weyl}\\ \text{fermions}}} R[\psi_i] \, T(r_i)$

$$= 2 \cdot \underset{\substack{\uparrow \\ \text{R-charge of vector multiplet Weyl fermion}}}{1} \cdot T(adj) + 2 \cdot (-1) \cdot \sum_{\substack{\text{hypers}\\ i}} T(r_i)$$

$$= 2 \left(T(adj) - \sum_{\substack{\text{hypers}\\ i}} T(r_i) \right)$$

Vector multiplet → 2 Weyl fermions + gaugino + scalar (see pg 25)

hypermultiplet → 2 Weyl fermions + 2 scalars (see pg 26)

Sub-example : $SU(N_c)$ gauge group with N_f hypermultiplets in the fundamental representation

anomaly coefficient $\quad A = 2 (\underbrace{2N_c}_{\text{Dynkin index of adjoint}} - N_f)$

N_f: +1 = Dynkin index of the fundamental

50:35

So, due to the $U(1)_R$-symmetry anomaly, under a $U(1)_R$ transformation with parameter α

the action becomes

$$S \longrightarrow S + \alpha A \underbrace{\int d^4x \frac{1}{16\pi^2} \mathrm{tr}_{\text{fund}} (F_{\mu\nu} {}^*F^{\mu\nu})}_{\substack{\text{instanton number} \\ K}}$$

So, we can "restore" $U(1)_R$ symmetry by declaring that under this $U(1)_R$-transformation, not only will the Weyl fermions transform as

$$\psi_i \longmapsto \exp\left[i\alpha R[\psi_i]\right] \psi_i$$

but the θ-angle also transforms as

$$\theta \longmapsto \theta + \alpha A$$

Note that changing θ changes the coupling so this isn't really a symmetry.

- But we know that θ is really an angle, so phys should be invariant under $\theta \mapsto \theta + 2\pi$.

So if αA is an integer multiple of 2π, you do get a symmetry.

$\rightarrow$ If $\alpha = \frac{2\pi n}{A}$, $n \in \mathbb{Z}$, you do really have a symmetry.

$\therefore$ $U(1)_R$-symmetry of the classical action is broken by anomaly to a discrete subgroup $\mathbb{Z}_A$

$$U(1)_R \xrightarrow{\text{anomaly}} \mathbb{Z}_A$$

$\mathbb{Z}_A$ $\rightarrow$ consists of $U(1)_R$-symmetry rotations with discrete parameter $\alpha = 2\pi n / A$.

(Student Comment: there is a relation of this to gaugino condensation.)

- PERTURBATIVE GAUGE COUPLING ; RENORMALIZATION

1-loop $\quad \mu \frac{\partial g}{\partial \mu} = -\frac{b}{16\pi^2} g^3$ (b: one-loop β-fxn coefficient)

Complexified gauge coupling $\quad \tau(\mu) = \frac{\theta(\mu)}{2\pi} + i \frac{4\pi}{g^2(\mu)}$

- can think of μ as a complex variable

1-loop running is governed by

$$\boxed{\left(\frac{\Lambda}{\mu}\right)^b = \exp\left(-\frac{8\pi^2}{g(\mu)} + i\theta(\mu)\right) = \exp\left(2\pi i \tau(\mu)\right)}$$

$$\left(\frac{\Lambda}{\mu}\right)^b = e^{-S^{\text{BPS}}_{\text{inst}}}$$

(one-instanton action for BPS states)

take log $\rightarrow$ complexified coupling runs logarithmically with energy scale

Specializing to $\mathcal{N}=2$ gauge theory with some gauge group and matter content,

$$b = \underbrace{T(\text{adj})}_{\text{contrib. from vector multiplet}} - \sum_{\text{hypers } i} T(r_i) = \frac{A}{2}$$

(A = anomaly coefficient defined earlier for $\mathcal{N}=2$)

The boxed one-loop formula, by the usual holomorphy argument, is exact to all orders in perturbation theory. But it does receive instanton corrections.

Let us compute the R-charge of Λ^b. This is the R-charge of the 1-instanton action S^{BPS}_{inst}.

$$S^{BPS}_{inst} = \frac{8\pi^2}{g^2(\mu)} - i\theta(\mu)$$

$$R[\Lambda^b] = R[e^{i\theta}] \overset{\text{how?}}{=} A = 2b$$

(the gauge coupling does not transform under R-symmetry, but the θ-angle does transform because of the $U(1)_R$-anomaly)

$$\Rightarrow \boxed{R[\Lambda] = 2}$$

- μ, the energy scale, can be thought of as sitting in an $\mathcal{N}=1$ chiral multiplet
- Λ appears on the same footing; Λ is really living in some $\mathcal{N}=1$ vector or chiral multiplet
- Λ is viewed as a background scalar in an $\mathcal{N}=1$ vector multiplet and it also has a $U(1)_R$-charge $+2$. chiral?
- Λ is also an $SU(2)_R$-singlet. ($SU(2)_R$ symmetry does not have an anomaly so the θ angle does not transform).

→ Λ is "like a" background scalar in the $\mathcal{N}=2$ vector multiplet?

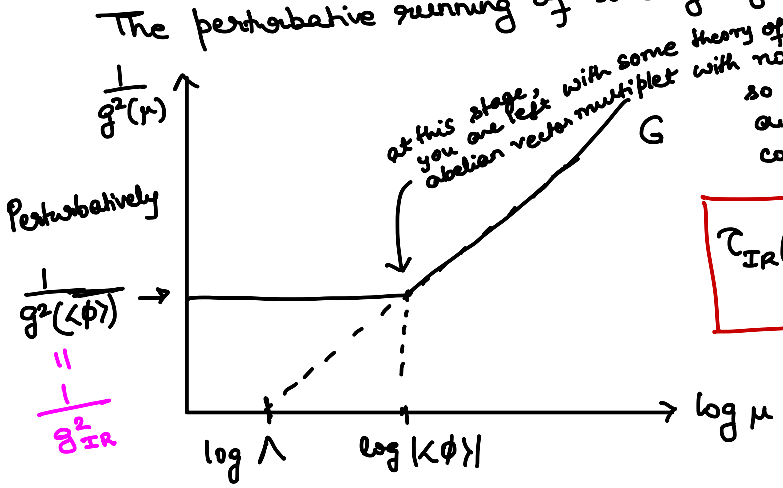
Answer: Yes

JUSTIFICATION

Consider a vacuum on the Coulomb Branch where the gauge group G is broken to $U(1)^{rk(G)} (\rtimes W(G))$.

$$\underset{(UV)}{G} \xrightarrow{\langle\phi\rangle} U(1)^{rk(G)} \rtimes W(G)$$

The perturbative running of the gauge coupling looks as follows:



$$\boxed{\tau_{IR}(\langle\phi\rangle) = \frac{1}{2\pi i}\, b \log \frac{\Lambda}{\langle\phi\rangle}}$$

Λ appears on the same footing in this formula as $\langle\phi\rangle$, which sits in an $\mathcal{N}=2$ vector multiplet. So from this formula, we see that Λ sits in the $\mathcal{N}=2$ vector multiplet. → using this conclude that metric on Higgs branch does not receive corrections due to Λ

Using holomorphy, one can conclude that there are no extra perturbative corrections to the formula $\left(\frac{\Lambda}{\mu}\right)^b = e^{-S^{BPS}_{inst}}$ beyond one-loop, essentially because these corrections — if present — will violate holomorphy.

Higher order perturbative corrections will look like

$$\text{loop corrections}: \quad g^{2n} \sim \left(\log|\Lambda|\right)^n \qquad \begin{array}{c}\text{not holomorphic}\\ \downarrow \\ \text{not allowed}\end{array}$$

On the other hand, one <u>can</u> have instanton corrections; they will look like:

$$k\text{-instanton correction}: \left(\frac{\Lambda}{\langle\phi\rangle}\right)^{bk} \longrightarrow \begin{array}{c}\text{holomorphic}\\ \Downarrow \\ \text{allowed.}\end{array}$$

And indeed, instanton corrections appear.

<u>Exercise</u>: Consider an $SU(2)$ gauge theory with matter in the fundamental representation, given an isospin. Determine the matter content such that the 1-loop β-function is zero. In this case, the theory is <u>CONFORMAL</u>.

PART 3: SEIBERG-WITTEN SOLUTION OF THE IR DYNAMICS OF $\mathcal{N}=2$ SU(2) SUPER YANG MILLS (SYM)

Seiberg & Witten: Determine exact IR effective action of pure SU(2) $\mathcal{N}=2$ SYM

- HOLOMORPHY
- R-SYMMETRY
- UNITARITY
- ELECTRIC-MAGNETIC DUALITY
- WEAK COUPLING RESULT

+ ... a bit of mathematics ☺

A crucial role will be played by an auxiliary Riemann surface (elliptic curve) which will contain all this information.

UV LAGRANGIAN FOR SU(2) SYM (for future reference)

$$\mathcal{L}_{SU(2)\,SYM} = \frac{1}{4\pi}\,\mathrm{Im}\left[\tau^{SU(2)}\,\mathrm{tr}\Big(\underbrace{\int d^2\theta\, W^\alpha W_\alpha}_{\text{gaugino kinetic terms}} + \underbrace{\int d^2\theta\, \Phi e^V \Phi}_{\text{kinetic terms for the scalar}}\Big)\right]$$

$$\tau^{SU(2)} = \frac{\Theta^{SU(2)}}{2\pi} + i\,\frac{4\pi^2}{g^2_{SU(2)}} \longrightarrow \text{coupling in the UV SU(2) theory}$$

scalar potential $\rightarrow V = \dfrac{1}{g^2_{SU(2)}}\,\mathrm{Tr}\,[\phi,\phi^+]^2$ (due to D-terms)

minimized by $\Phi = a\,\sigma_z = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ (solution to D-term equations)

up to gauge transformations

$\{a \in \mathbb{C}\}/\mathbb{Z}_2$ Weyl: $a \to -a$

breaks $SU(2) \longrightarrow U(1)$

gauge-invariant coordinate on the Coulomb branch is the Casimir $u = \frac{1}{2}\mathrm{Tr}\,\phi^2 \overset{\text{classically}}{\equiv} a^2$

Focus on massless fields

$$F_{\mu\nu} \equiv F_{\mu\nu}^{U(1)} = \begin{pmatrix} F_{\mu\nu}^{U(1)} & 0 \\ 0 & -F_{\mu\nu}^{U(1)} \end{pmatrix}$$

similarly for superpartners

$$\mathcal{L}_{SU(2)\,SYM}\Big|_{\text{massless fields}} = \mathcal{L}_{U(1)\ \text{Super Maxwell}} \longrightarrow \text{IR lagrangian for massless fields (classically)}$$

$$\underset{\text{IR coupling}}{\tau^{U(1)}} = 2\,\underset{\text{UV coupling}}{\tau^{SU(2)}}$$

This was at the classical level.

What changes at the quantum level?

Quantum

- quantum v.e.v. for ϕ

$$\langle \phi \rangle = a\,\sigma_3$$

R-charge : 2

- $u \equiv \left\langle \frac{1}{2} \operatorname{Tr} \phi^2 \right\rangle = a^2 + \underbrace{\cdots\cdots}_{\text{quantum corrections}}$ ← R-charge : 4

1:24:05

$$\underset{e^{i\alpha}}{U(1)_R} \xrightarrow{\text{anomaly}} \underset{e^{i\frac{2\pi n}{8}}}{\mathbb{Z}_8} \xrightarrow[\text{spont.}]{\langle \operatorname{Tr}\phi^2\rangle} \underset{e^{i\frac{2\pi m}{4}}}{\mathbb{Z}_4}$$

$n = 0, 1, \ldots, 7$ $\qquad$ $m = 0, 1, 2, 3$

$u \mapsto e^{4i\alpha} u$ (as u has R-charge 4)

$\alpha = \frac{2\pi n}{8}$ so $u \mapsto e^{i\pi n} u$

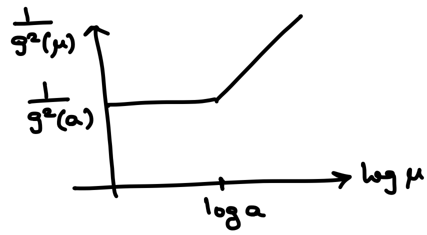
$\alpha = \frac{2\pi m}{4}$, $u \mapsto e^{i2\pi m} u$ or $u \mapsto u$

$\mathbb{Z}_4$ subgroup survives even in the vacuum where ϕ^2 acquires a v.e.v

One-loop running of the coupling

$$\left(\frac{\Lambda}{\mu}\right)^4 = e^{2\pi i \tau^{SU(2)}(\mu)}$$

$$\text{UV :} \quad \left(\frac{\Lambda}{\Lambda_{UV}}\right)^4 = e^{2\pi i \tau^{SU(2)}(\Lambda_{UV})} = e^{2\pi i \tau_{UV}}$$ ← UV coupling



Let $\tau(a)$ be the IR effective coupling $\tau^{U(1)}_{\text{eff}}$

$$\begin{aligned}\tau(a) &\cong 2\tau^{SU(2)}(a) \\ &= 2\tau_{UV} - \frac{8}{2\pi i} \log \frac{a}{\Lambda_{UV}} + \cdots \\ &= -\frac{8}{2\pi i} \log \frac{a}{\Lambda} + \cdots\cdots\end{aligned}$$

When $|a| \gg |\Lambda|$, instanton corrections will be small and this approx. will be a good approx.

N=2 IR EFFECTIVE ACTION

U(1) N=2 vector multiplet

$\mathcal{A} = (A, W_\alpha)$

$A = a + \cdots$

$W_\alpha = \lambda_\alpha + \cdots$

$$\boxed{\begin{aligned}\mathcal{L}_{eff} &= \frac{1}{4\pi}\int d^4\theta\, \mathcal{F}(\mathcal{A}) \\ &= \frac{1}{4\pi}\mathrm{Im}\left(\int d^2\theta\, \mathcal{F}'(A)\bar{A} + \int d^2\theta\, \frac{1}{2}\mathcal{F}''(A)W^\alpha W_\alpha\right)\end{aligned}}$$

$\rightarrow$ see pg. 31 for a refresher

KÄHLER POTENTIAL

$$\boxed{K(A,\bar{A}) = \frac{1}{4\pi}\mathrm{Im}\left(\mathcal{F}'(A)\bar{A}\right)}$$

$\rightarrow$ not covariant under all coordinate changes of A & $\bar{A}$. We are using so called a "special coordinates". In arbitrary coordinates the two terms in $\mathcal{L}_{eff}$ are not related so simply to each other. This makes $\mathcal{N}=2$ supersymmetry manifest.

METRIC ON THE COULOMB BRANCH $\mathcal{M}_{coul}$

$\downarrow$

nothing but the Kähler metric that follows from this Kähler potential

$$\boxed{ds^2 = \frac{1}{4\pi}\mathrm{Im}\, \mathcal{F}''(a)\, da\, d\bar{a}}$$

What is special about these coordinates $\rightarrow$ 'a' is related to the gaugino superfields linearly.

EFFECTIVE COMPLEXIFIED GAUGE COUPLING

$$\boxed{\tau(a) = \mathcal{F}''(a)}$$

so the metric on the Coulomb branch can be written as

$$\boxed{ds^2 = \frac{1}{4\pi}\mathrm{Im}\, \tau(a)\, da\, d\bar{a}}$$

It may seem as if the action is determined by the first derivative and the second derivative of $\mathcal{F}(A)$ but in fact only the second derivative is physical: one can identify $\mathcal{F}$ up to linear terms

$$\mathcal{F} \sim \mathcal{F} + \alpha a + \beta$$

This does not change the action: this is obvious for the constant term β. What about the linear term? This shifts the Kähler potential by some term proportional to the imaginary part of $\bar{A}$: this is just a Kähler transformation. It doesn't do anything to the low-energy physics. The coefficient of $da\,d\bar{a}$ (which is $|da|^2$, which is positive) is $\frac{1}{4\pi}\mathrm{Im}(\tau(a))$, which must be positive (as $ds^2 > 0$). This requires

$$\boxed{\mathrm{Im}\,\tau(a) > 0}$$

UNITARITY

Since this metric is the Kähler metric which determines the kinetic term, this condition is just the physical requirement of UNITARITY.

We can compute the prepotential $\mathcal{F}(a)$ perturbatively using the 1-loop running of the coupling. This is because we know that the second derivative of the prepotential is given by the effective coupling ($\tau(a) = \mathcal{F}''(a)$).

$$\boxed{\mathcal{F}_{pert}(a) = \iint \tau(a) = \frac{i}{\pi} a^2 \left(\log \frac{a^2}{\Lambda^2} - 3\right)}$$

from the 1-loop formula for the gauge coupling

Non-perturbatively, in the full theory, the full exact effective prepotential

$$\boxed{\mathcal{F}(a) = \frac{i}{\pi} a^2 \left(\log \frac{a^2}{\Lambda^2} - 3\right) + \underbrace{\sum_{k=1}^{\infty} f_k \left(\frac{\Lambda}{a}\right)^{4k} a^2}}$$

INSTANTON CORRECTIONS

$|a| \gg |\Lambda|$ (weak coupling region)
the answer better agree with very good accuracy with the perturbative answer
$\Rightarrow$ reason for only +ve powers

By holomorphy any holomorphic function of $\left(\frac{\Lambda}{a}\right)^4$ could appear.

The 4 is just the 1-loop β-fxn coefficient

The prepotential must carry charge 4, which is why f_k multiplies an a^2.

In principle, we would like to know all these coefficients.

$$\tau(a) = -\frac{2}{2\pi i} \log \frac{a^4}{\Lambda^4} + \sum_{k=1}^{\infty} \tau_k \left(\frac{\Lambda}{a}\right)^{4k}$$

perturbative part
+
instanton corrections

Define "a-dual":

$$\boxed{a_D = \mathcal{F}'(a)}$$

now $K = \frac{1}{4\pi} \operatorname{Im}(F'(a)\bar{a})$

so,

$$\boxed{K = \frac{1}{4\pi} \operatorname{Im}(a_D \bar{a})}$$

$$\boxed{ds^2 = \frac{1}{4\pi} \operatorname{Im}(da_D \, d\bar{a})}$$

$$\boxed{\tau(a) = \frac{\partial a_D}{\partial a}}$$

From this form of the equations, we see that a and a_D appear on the same footing in the Kähler potential and in the metric on the moduli space.

This is in fact related to electric-magnetic duality.

Sometimes people call $\begin{pmatrix} a_D \\ a \end{pmatrix}$ <u>special coordinates</u>.

Geometrically, the Coulomb Branch of the moduli space is a Kähler manifold. Here we have some extra structure where the Kähler potential takes this very simple form $\left(K = \frac{1}{4\pi} \mathrm{Im}(a_D \bar{a})\right)$. Kähler manifolds with this structure are called RIGID SPECIAL KÄHLER MANIFOLDS, meaning that we can introduce coordinates a and a_D so that the Kähler potential and metric take the abovementioned forms.

So, the geometry of the Coulomb Branch is a Rigid Special Kähler manifold.

Let us look at the Kähler potential $K = \frac{1}{4\pi} \mathrm{Im}(a_D \bar{a})$

and the Kähler metric $ds^2 = \frac{1}{4\pi} \mathrm{Im}(da_D \, d\bar{a})$

- The Kähler metric is invariant under the following change of variables

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto M \begin{pmatrix} a_D \\ a \end{pmatrix} + \text{const. vector} \; ; \; \tau(a) \to \frac{a\tau + b}{c\tau + d}$$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det(M) = 1$

where $M \in SL(2, \mathbb{R})$. This is reminiscent of electric-magnetic duality where we had an $SL(2, \mathbb{Z})$.

- We will see that these transformations are indeed electric-magnetic duality transformations and when we impose the condition that charges are quantized, the $M \in SL(2, \mathbb{Z})$ as we saw in the non-supersymmetric case earlier.

ELECTRIC-MAGNETIC DUALITY

$SL(2, \mathbb{R})$ is generated by matrices $T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ where $b \in \mathbb{R}$

S-transformations $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$T_b : \tau \longmapsto \tau + b \; ; \text{ for now, } b \in \mathbb{R}$$

But physics is invariant only if $\tau \mapsto \tau + 1$.

$\Rightarrow \boxed{b \in \mathbb{Z}}$

So eventually, the $SL(2, \mathbb{R})$ group of metric invariant transformations that leave the metric invariant has to be restricted to $SL(2, \mathbb{Z})$.

Physics is invariant only under shifts of the θ angle by $2\pi \times$ (integer).

This is nothing but the $SL(2, \mathbb{Z})$ - duality group that we saw in the nonsupersymmetric case.

References

Treating coupling constants as background superfields

- Seiberg : 9309335
- Intriligator & Seiberg :

ELECTRIC – MAGNETIC DUALITY (in the supersymmetric context)

S-transformation

(A, V) — W_α

$$Z = \underbrace{\int \mathcal{D}A}_{\mathcal{N}=1 \text{ chiral}} e^{\overset{\text{does not depend on } V}{i \cdots}} \underbrace{\int \mathcal{D}V}_{\mathcal{N}=1 \text{ vector}} e^{\frac{i}{8\pi}\int d^4x \, \mathrm{Im} \int d^2\theta \, \tau(A) W^\alpha W_\alpha} \Big|_{W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V}$$

$$= \int \mathcal{D}A \, e^{i\cdots} \underbrace{\int \mathcal{D}V_D}_{\text{vector}} \underbrace{\int \mathcal{D}W_\alpha}_{\text{chiral}} e^{\left[\frac{i}{8\pi}\int d^4x \, \mathrm{Im}\int d^2\theta \, \tau(A) W^\alpha W_\alpha + \frac{i}{4\pi}\int d^4x \, \mathrm{Im}\int d^4\theta \, V_D D_\alpha W^\alpha\right]}$$

V_D: Lagrange multiplier → enforces the supersymmetric version of the Bianchi identity, which is $\boxed{\mathrm{Im}(D_\alpha W^\alpha) = 0}$

If we do the V_D integral first, this enforces the Bianchi identity $\mathrm{Im}\, D_\alpha W^\alpha = 0$ and since we are working on $\mathbb{R}^4$ where there is no topology, the only solution is that $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$, and hence we're back to the original form of the path integral. But lets do the W_α integral first.

$$\int d^4\theta \, V_D D_\alpha W^\alpha = -\int d^4\theta \, (D_\alpha V_D) W^\alpha$$
$$= ?\int d^2\theta \, d^2\bar{\theta} \, (D_\alpha V_D) W^\alpha$$
$$= ?\int d^2\theta \, \bar{D}^2\left((D_\alpha V_D) W^\alpha\right)$$
$$= \int d^2\theta \left(\bar{D}^2 (D_\alpha V_D)\right) W^\alpha \quad \text{as } \bar{D}_{\dot\alpha} W^{\dot\alpha} = 0$$
$$= -4\int d^2\theta \, (W_D)_\alpha W^\alpha \Big|_{(W_D)_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V_D}$$

? → fix sign

So now we have, in the exponent, a quadratic term in W and a linear term in W. So, we can complete the square, and do the Gaussian integral over W_α

$$\rightsquigarrow Z = \int \mathcal{D}A \, e^{i\cdots} \int \mathcal{D}V_D \, e^{\frac{i}{8\pi}\int d^4x \, \mathrm{Im}\int d^2\theta \left(-\frac{1}{\tau(A)}\right)(W_D)^\alpha (W_D)_\alpha} \Big|_{W_D = -\frac{1}{4}\bar{D}^2 D V_D}$$

$-\frac{1}{\tau(A)} \equiv \tau_D(A_D)$ (we'll see this on the next page)

So what we have done is replace the original $\mathcal{N}=1$ vector superfield V with the dual $\mathcal{N}=1$ vector superfield V_D (and hence get a dual YM action), with a dual coupling τ_D.

This was the dualization of the $\mathcal{N}=1$ vector V that sits inside the $\mathcal{N}=2$ vector multiplet (A, V). Now we also want to dualize A.

It turns out that the dualization of A, the scalar in the vector multiplet, is much easier. This is related to the Rigid Special Kähler geometry introduced earlier.

(A) Recall that we have introduced a coordinate dual to A, which was the derivative of the prepotential w.r.t. A. The kinetic term for $\bar{A}$ & A just depended on A and this dual coordinate.

Consider the transformation to dual variables

$$\begin{pmatrix} \frac{\partial \mathcal{F}(A)}{\partial A} \\ A \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \mathcal{F}_D(A_D)}{\partial A_D} \\ A_D \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{S\text{-transformation}} \begin{pmatrix} \frac{\partial \mathcal{F}(A)}{\partial A} \\ A \end{pmatrix}$$

$$= \begin{pmatrix} -A \\ \frac{\partial \mathcal{F}}{\partial A} \end{pmatrix}$$

so in particular, we want

$$\boxed{\begin{aligned} A_D &= \mathcal{F}'(A) \\ A &= -F_D'(A_D) \end{aligned}}$$

Recall Legendre transformations. The second eqn is just the inverse Legendre transformation of the first.

Legendre Transform

$$\boxed{\mathcal{F}_D(A_D) = -A_D A + \mathcal{F}(A)}$$

Dual effective Coupling

$$\tau_D(a_D) = \mathcal{F}_D''(a_D) = -\frac{\partial a}{\partial a_D} = -\left(\frac{\partial a_D}{\partial a}\right)^{-1}$$

$$= -\left(F''(a)\right)^{-1} = -\frac{1}{\tau(a)}$$

$$\Rightarrow \boxed{\tau_D(a_D) = -\frac{1}{\tau(a)}}$$

Since this S-transformation looks very much like a canonical transformation, we expect the Jacobian for the change of variables to be unity. So, the path integral measure is invariant under this transformation.

The kinetic term for the chirals (denoted as ... in the argument of the first exponent above) is

$$\text{Im} \int d^4\theta \, \mathcal{F}'(A)\,\bar{A}$$

This was the Kähler potential term. Now,

$$\text{Im} \int d^4\theta \, \mathcal{F}'(A)\,\bar{A} = -\,\text{Im} \int d^4\theta \, A_D \, \overline{\mathcal{F}_D'(A_D)}$$

$$= +\,\text{Im} \int d^4\theta \, \mathcal{F}_D'(A_D)\,\bar{A}_D \longrightarrow$$

so we end up with a dual kinetic term which looks exactly like the original.

So the original path integral over the $\mathcal{N}=2$ multiplet variables can be rewritten in terms of dual variables.

Next, we discuss the central charge formula.

Central Charge Formula

- Georgi–Glashow Model (no SUSY): $\boxed{M \geqslant |a|\,|n + m\,\underbrace{\tau^{U(1)}}|}$ Classically

electric charge

magnetic charge

inequality satisfied by all states; BPS states saturate it (=)

vev for the scalar

effective coupling for massless vector fields

- $\mathcal{N}=2$ SUSY: $\boxed{M \geqslant |Z|}$

central charge

bound is saturated by short or BPS multiplets

BPS equations follow from the required supersymmetry

A BPS state stays BPS $M = |Z|$. States cannot come out of nowhere. (M & $|Z|$ receive identical quantum corrections)

▶ We are interested in a (quantum) exact formula for the central charge of BPS states in terms of the vev, electric and magnetic charges and coupling, like we had for the Georgi-Glashow model (non-supersymmetric case).

Seiberg & Witten's argument

Hypermultiplet of electric charge n: couples to $\mathcal{N}=1$ chiral in vector multiplet

$$W = n A\, Q\tilde{Q}$$

(follows from $\mathcal{N}=2$ SUSY)

Z

$\mathcal{N}=1$ abelian scalar (of the low-energy theory)

$$Z = n a$$

By S-transf. of E-M duality, we expect that for a monopole of magnetic charge m, the central charge will look like $Z = m\, a_D$ (a_D = dual scalar)

By linearity, Seiberg & Witten proposed the following formula for the central charge

$$\boxed{Z = m a_D + n a \quad = (m, n) \begin{pmatrix} a_D \\ a \end{pmatrix}}$$

Some comments on the central charge formula

1. Correct (semi)classical limit
2. $SL(2,\mathbb{Z})$ duality invariant

$$(m, n) \longmapsto (m, n) M^{-1} \qquad M \in SL(2,\mathbb{Z})$$

earlier we showed that special coordinates transformed by M (rather than M^{-1}):

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto M \begin{pmatrix} a_D \\ a \end{pmatrix}$$

$$\therefore \; Z = (m, n) \begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto (m, n) M^{-1} M \begin{pmatrix} a_D \\ a \end{pmatrix} = Z$$

Note: Earlier in the context of Rigid Special Kähler Geometry, we said that the Kähler potential in the low energy effective action is invariant under a more general transformation of the form

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto \underset{\substack{\uparrow \\ \in SL(2,\mathbb{C})}}{M} \begin{pmatrix} a_D \\ a \end{pmatrix} + \text{"constant"}$$

So we can ask: given this transformation, can we find a suitable transformation of (m, n) such that $Z = m a_D + n a$ is invariant?

The answer is NO. There is no way to compensate for this shift by a constant by a transformation of (m, n). $\Rightarrow$ "constant" $= 0$. Note that this statement is true only for gauge theories with no matter fields. For theories with matter fields, the central charge formula is modified: one has in addition to $(m a_D + n a)$ some terms that are linear in the masses; in such cases, the "constant" is nonzero. We will not discuss them.

Note: The entries of M have to be integers (i.e. M has to lie in $SL(2,\mathbb{Z})$ instead of $SL(2,\mathbb{R})$) because it acts on charges (m, n) which are quantized.

It is worth stressing here that the electric-magnetic duality is not a symmetry, and should really be viewed as a change in reference frame. But it is still quite useful as the S-transformation maps strong coupling to weak coupling.

- Rigid Special Kähler (RSK) geometry
- E-M Duality group ($SL(2,\mathbb{Z})$)
- Weak-Coupling results

$\longrightarrow$ Determine IR Effective Action on $\mathcal{M}_{coulomb}$

WEAKLY COUPLED REGION OF MODULI SPACE

$|a| \gg |\Lambda|$

$|a|$: vev of scalar in vector multiplet

$|\Lambda|$: strong coupling scale

$\longrightarrow$ then roughly speaking, the running coupling is frozen at the very high scales so we're starting with some asymptotically free theory which at high energies is weakly coupled.

Effective prepotential

$$\mathcal{F}(a) = \frac{i}{\pi} a^2 \left(\log \frac{a^2}{\Lambda^2} - 3 \right) + \sum_{K=1}^{\infty} f_K \left(\frac{\Lambda}{a} \right)^{4K} a^2$$

$$a_D = \mathcal{F}'(a) \cong \frac{2i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} - 2 \right) + \text{nonperturbative corrections}$$

Recall that 'a' is not a gauge invariant variable, but we have a gauge invariant variable 'u':

$$u = \left\langle \tfrac{1}{2} \mathrm{Tr}\, \phi^2 \right\rangle \cong a^2 + \text{instanton corrections} \quad (\text{for large } a)$$

u: good coordinate on the Coulomb Branch

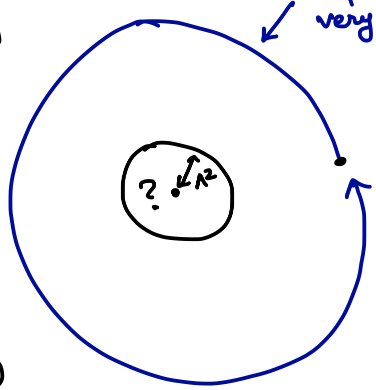
We can write everything in terms of the 'good coordinate' u.

$$\boxed{a_D = \frac{2i}{\pi} \sqrt{u} \left(\log \frac{u}{\Lambda^2} - 2 \right) + \cancel{\text{instanton corrections}}}$$

$\leftarrow$ dropped for perturbative corrections (large u, see below)

$$\boxed{a \cong \sqrt{u}}$$

So the Coulomb Branch is parametrized by u



loop at very large u

$\lfloor u$

$\mathcal{M}_{coulomb}$

Think of starting at a large u and adiabatically go around the blue loop (i.e. adiabatically vary u) until you reach the same point.

If the loop is v. large, we can use the weakly coupled approximation (perturbative approximation) for the special coordinates

"$u \longrightarrow e^{2\pi i} u$" $\longrightarrow$ What happens to a & a_D ?

- Deep within the Coulomb Branch, near the origin there will be a region of size of $O(\Lambda^2)$ where the interactions will be strongly coupled and we don't know yet what will happen there
- But if $u \gg \Lambda^2$, we can use the weakly coupled approximation

"$u \longrightarrow e^{2\pi i} u$"

$a \cong \sqrt{u}$ $\longrightarrow$ $e^{2\pi i}\sqrt{u} = -\sqrt{u} = -a$

$a_D \cong \frac{2i}{\pi}\sqrt{u}\left(\log\frac{u}{\Lambda^2} - 2\right)$ $\longrightarrow$ $-\frac{2i}{\pi}\sqrt{u}\left(\log\frac{u}{\Lambda^2} + 2\pi i - 2\right)$

$= -a_D + 4a$ (in terms of the original special coordinates)

"original" special coordinates

- As we go full circle counterclockwise in the $\mathbb{u}$ plane, $\begin{pmatrix} a_D \\ a \end{pmatrix}$ transforms as follows

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}}_{\text{MONODROMY MATRIX } M_\infty} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

MONODROMY AT ∞

(Multivalued $\begin{pmatrix} a_D \\ a \end{pmatrix}$.)

for spl. coords a_D & a associated to a loop at a very large u-value

- So, 'a_D' and 'a' are multi-valued functions; because of the logarithm and square root that appear in their definition, there are some Branch Cuts.

- This monodromy transformation should be contrasted with electric-magnetic duality. Electric-magnetic duality transformations are just passive transformations (i.e. changes of reference frame). The Monodromy transformation is an active transformation: we adiabatically changed the value of $u = \langle \frac{1}{2} \mathrm{Tr}\, \phi^2 \rangle$ as we went full circle around.

- Monodromy transformation; $u \longrightarrow e^{2\pi i} u$ is an active physical transformation.

- In particular, this monodromy transformation at ∞, changes the central charge. (Central charge is a physical observable as it determines the mass, therefore this is an active physical transformation).

- Alternatively, one could attribute the monodromy to the electric and magnetic charges (m, n) rather than to the special coordinates, i.e. say

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longmapsto \begin{pmatrix} a_D \\ a \end{pmatrix}$$

but $(m, n) \longmapsto (m, n) M_\infty = (-m, 4m-n)$

It is up to you what description you use, but the central charge will change depending on the description.

More on the monodromy matrix

- $M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \equiv PT^{-4}$ where $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ← can be thought of as charge conjugation

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

"Why P?": P is already present classically.

* Classically, $a = \sqrt{u}$
  and $a_D = \tau a$

The U(1) field strength is $F_{\mu\nu}^{U(1)} = \frac{1}{2a} \mathrm{Tr}\,(\phi F^{\mu\nu})$. So under monodromy since $a \to -a$, the U(1) field strength changes sign, so the magnetic charges change sign.

Why T^{-4}? • 1-loop monodromy, due to the 1-loop β-fxn coefficient

$b = 2N_c = 4$ ← same as the 4 in T^{-4}

coefficient of the log

• can also be interpreted as being due to the Witten effect.

Witten effect

as there is a shift of the electric charge by a multiple of the magnetic charge due to the Θ angle

$\left((m,n) \mapsto (m,n) M_\infty = (-m, 4m-n) \right)$

$$\tau(a) = -\frac{8}{2\pi i} \log \frac{a}{\Lambda} \longrightarrow \tau(a) - 4$$

⇑ due to

$$\Theta(a) \longrightarrow \Theta(a) - 8\pi$$

$4 \cdot 2\pi$ ← due to T^{-4}

▶ $u = \infty$ is a singularity, being the branch point of the logarithm. As the branch cut must end somewhere, there has to be another singularity — in the strongly coupled region.

Note that our analysis so far of the monodromy has been based on perturbative expressions (the 1-loop running of gauge coupling, and the square root). We have neglected nonperturbative instanton corrections. One might think they may also contribute to the monodromy. In fact, using the general form of the instanton corrections that we wrote above, we can argue that regardless of the coefficients in the instanton corrections, they will <u>not</u> affect the monodromy. The 1-loop running of gauge coupling fixes the monodromy at ∞ completely.

As stated above, we expect a second singularity in the strongly coupled region, where the perturbative approximation does not hold.

SINGULARITIES & MONODROMIES IN THE STRONGLY COUPLED REGION ($u \sim \Lambda^2$)

1) How many singularities? (How many singularities are there in the strongly coupled region in addition to the singularity at ∞?)

2) What are their monodromies?

3) What is the physical interpretation of the monodromies and singularities?

[The physical interpretation of the monodromy at ∞ is just the 1-loop running of gauge coupling.]

Here it is important to keep in mind the R-symmetry group of the problem. Recall that the $U(1)_R$-symmetry is classically broken by an anomaly to $\mathbb{Z}_8$ R-symmetry which is broken in turn by $u = \langle \frac{1}{2} \mathrm{Tr}\,\phi^2 \rangle$ to some $\mathbb{Z}_4$ subgroup.

$$U(1)_R \xrightarrow{\text{anomaly}} \mathbb{Z}_8 \xrightarrow{u = \langle \frac{1}{2}\mathrm{Tr}\,\phi^2 \rangle} \mathbb{Z}_4$$

In particular there is a $\mathbb{Z}_2$ subgroup (essentially the quotient of the $\mathbb{Z}_8$ by the $\mathbb{Z}_4$) which is broken. The nontrivial transformation in this $\mathbb{Z}_2$ is

$$u \longrightarrow -u$$

So essentially physics at any point u should be equivalent to the physics at $-u$. So in particular if there is a singularity at u, there must be another singularity at $-u$.

– SIMPLEST OPTION: single extra singularity, which can only be at $u = 0$ (if it is at $u = u_0 \neq 0$, there'll be another singularity at $u = -u_0 \neq 0$!)

In this case, we'll have two singularities in total: one at ∞ and one at 0.

This looks like the best guess. Classically $u = 0$ is like $a = 0$.

In particular you don't get back to the scalar field. The gauge group $SU(2)$ is not broken so we'll have massless gauge bosons at $u = 0$ for the whole $SU(2)$. The fact that we have extra massless gauge bosons leads to a singularity of the low-energy effective action.

The central charge formula was $Z = a(n + m\tau^{U(1)})$ classically.

Gluons are massless. All dyons in this formula are massless. So, there are all sorts of massless states that we did not take into account in our low-energy effective action. One can expect these to be a source of the singularity.

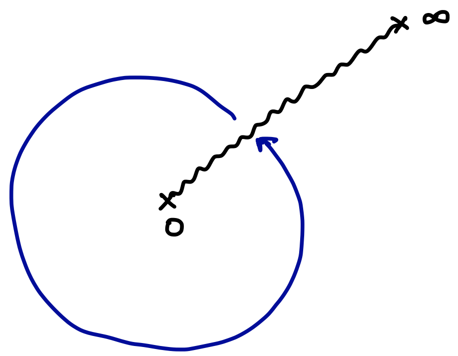
This was just a classical analysis.

Quantum mechanically, $u = 0 \Rightarrow$ strong coupling (so we cannot use the classical intuition and classical formulas to derive any conclusion)

And in fact there is a simple argument that excludes the "simple option" of having a single extra singularity at $u = 0$.

This option is excluded.

In fact there is no point in the Coulomb Branch where one gets massless gluons.
The point $a=0$ is not on the quantum moduli space.



$M_\infty = M_0$

a^2 is not affected by the monodromy.
$\Rightarrow$ good Global coordinate on $\mathcal{M}_{\text{cou}}$ if this situation is realized.

$$a_D = \frac{2i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} - 2 \right) + a \underbrace{g(a^2)}$$ Entire function of a^2

$$a = \sqrt{a^2}$$

The effective coupling is

$$\tau = \frac{\partial a_D}{\partial a} = - \frac{4}{2\pi i} \log \left(\frac{a^2}{\Lambda^2} \right) + \underbrace{f(a^2)}$$ another entire function of a^2

$$f(a^2) = g(a^2) + 2a^2 g'(a^2)$$

We are interested in $\operatorname{Im}\tau$ and want to ensure that $\operatorname{Im}\tau > 0$ for unitarity.

$\operatorname{Im}\tau$ is a harmonic function
$\hookrightarrow$ τ is holomorphic $\Rightarrow$ $\bar\partial \tau = 0 \Rightarrow \partial\bar\partial \operatorname{Im}\tau = 0$

$\therefore$ $\operatorname{Im}\tau$ does not have a minimum somewhere $\rightarrow$ unitarity is violated
$\rightarrow$ must become negative somewhere
$\Rightarrow$ the possibility that there is a single singularity violates unitarity.

SINGULARITY : is a signal of the breakdown of the IR EFFECTIVE ACTION.
It happens when there are a extra MASSLESS DEGREES OF FREEDOM.

- Which kind of particles or solitons can become massless? [read Seiberg & Witten's papers]
  - Gluons (W-bosons) $\longrightarrow a = 0$
  - Massless non-Abelian gauge fields
    - expect ASYMPTOTIC CONFORMAL INVARIANCE (some nontrivial IR fixed point)
      must have $SU(2)_R \times U(1)_R$ symmetry

You need the full $U(1)$ R-symmetry. In the theory that we're studying, i.e. the pure $SU(2)$ gauge theory, classically there is an $SU(2) \times U(1)$ R-symmetry but the $U(1)_R$-symmetry is anomalous.
So if we imagine that there is an SCFT sitting at the point $a=0$ in moduli space, essentially there must be an accidental $U(1)_R$-symmetry group that does not "cancel". While this is in principle possible along the RG-flow,

its hard to imagine how this can happen. Seiberg and Witten strengthen their argument by considering other possibilities and arguing them out.

BPS MONOPOLES & DYONS? Yes (Monopoles & Dyons become massless)

In particular, BPS monopoles and dyons sit in hypermultiplets.

They are BPS, so their central & mass charge is given by

$$M = |Z| = |n a + m a_D|$$

(m, n) is massless $\longleftrightarrow$ $\boxed{n a + m a_D = 0}$

dyon with mag. charge m and electric charge n

[Note: a monopole with mag. charge 1 is denoted by (1, 0)].

Observation: Use $SL(2,\mathbb{Z})$ e-m duality + weak coupling results to compute monodromy matrix around a singularity where a (m, n) dyon becomes massless.

In particular we can take (m, n) to be coprime, i.e. $\gcd(m,n) = 1$. Then (m, n) are related by $SL(2,\mathbb{Z})$ to (0, 1). That is, we can transform to a reference frame where the dyon looks like a particle of electric charge 1.

▸ Go to dual frame where the dyon has dual charges $(m', n') = (0, 1)$.

Electric-magnetic duality transformation $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ (we use M now for the monodromy matrix) earlier we used M for the e-m duality matrix.

$$\det(A) = \alpha\delta - \beta\gamma = 1.$$

$$A^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

$$(0,1) \equiv (m', n') = (m, n) A^{-1} = (m, n) \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

$$= (m\delta - n\gamma \, , \, -m\beta + n\alpha)$$

$$\therefore \left.\begin{aligned} 0 &= m\delta - n\gamma \\ 1 &= -m\beta + n\alpha \end{aligned}\right\} \Rightarrow \text{fixes 2 of the entries in the matrix } A^{-1}$$

At the same time, since this is an e-m duality transformation, the special coordinates transform to dual special coordinates as

$$\begin{pmatrix} a_D' \\ a' \end{pmatrix} = A \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix}$$

In the dual frame: dual U(1) gauge theory with a massless hypermultiplet of dual electric charge 1 $\Leftrightarrow$ $\mathcal{N} = 2$ (dual) Super QED theory.

In particular, we can compute the running of the dual coupling.

$$\tau'(a') \cong -\frac{i}{2\pi}\log\left(\frac{a'}{\Lambda'}\right) \qquad \text{using the 1-loop } \beta\text{-fxn}$$

This (dual) SQED theory is free in the IR but runs to strong coupling at high energy, much like non-supersymmetric QED.

$\Lambda' \rightarrow$ Landau pole for dual SQED (the UV scale where the coupling diverges)

[Don't confuse Λ' with the dynamically generated strong coupling scale that we discussed in the context of qtm. corrections.]

In particular, recall that the complexified coupling is the second derivative of the prepotential, and the dual special coordinate is the first derivative.

$$\tau'(a') \xRightarrow[\text{once}]{\text{Integrate}} a'_D(a') = -\frac{ia'}{2\pi}\left(\log\left(\frac{a'}{\Lambda'}\right) - 1\right)$$

a' is a good local coordinate near the singularity where $u = u_0$

$$a' \cong c_0(u-u_0) + \cdots$$

One can repeat the following analysis and argue that the leading term is linear in $(u-u_0)$ and is not $(u-u_0)$ raised to some higher power.

$\underline{u|}$

Consider going around the singularity

Encircle u_0

$$(u-u_0) \longmapsto e^{2\pi i}(u-u_0)$$

Then,
$$a' \longrightarrow a' \quad \text{(does not pick up monodromy)}$$
$$a'_D \longrightarrow a'_D + a'$$
$$\therefore \quad \tau' \longrightarrow \tau' + 1$$

The monodromy around u_0 is

$$M'_{u_0} = M^{(0,1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{(in the dual frame)}$$

This tells us that if there is a point with electric charge 1 and magnetic charge 0 which is a massless hypermultiplet, then if we go around this point in moduli space, there is a monodromy, given by M_{u_0}.

<u>Exercise</u>. Use $SL(2,\mathbb{Z})$ e-m duality transformations to deduce that the matrix

$$M^{(m,n)} = \begin{pmatrix} 1+mn & n^2 \\ -m^2 & 1-mn \end{pmatrix}$$

is the monodromy matrix around a singularity due to a massless dyon (m,n).

In particular if $(m,n) = (0,1)$, we get back $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the matrix $M^{(0,1)}$ discussed above.

REMARK: Under monodromy (m,n), the charge vector itself transforms as

$$(m,n) \longmapsto (m,n)\, M^{(m,n)} = (m,n) \begin{pmatrix} 1+mn & n^2 \\ -m^2 & 1-mn \end{pmatrix}$$

$$= \left(m + m^2 n - m^2 n \,,\; mn^2 + n - mn^2\right)$$

$$= (m,n)$$

- The magnetic-electric charge vector of the dyon which becomes massless is a left-eigenstate of the matrix $M^{(m,n)}$
- The dyon responsible for the singularity is invariant under monodromy.

This had to be because this is the only way in which the statement, "the dyon is massless", makes sense. If it were to be that if one were to go around the singularity, the state would transform to one with different charges, it would not make sense to say that the state is massless.

Lecture 6 <some portion of the video seems to be missing>

Weak Coupling Singularity ($u=\infty$)

$$\mathcal{F}(a) = \frac{i}{\pi} a^2 \left(\log \frac{a^2}{\Lambda^2} - 3\right) + a^2 \sum_{k=1}^{\infty} f_k \left(\frac{\Lambda}{a}\right)^{4k}$$

$$u \longrightarrow e^{2\pi i} u \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

Massless (m,n) Dyon Singularity ($u=?$)

$SL(2,\mathbb{Z})$ rotate to DUAL FRAME $\quad (m', n') = (0,1)$

$$\mathcal{F}'(a') = -\frac{i}{8\pi} a'^2 \left(\log \frac{a'^2}{\Lambda^2} - 3\right) + a'^2 \sum_{n=1}^{\infty} g_n \left(\frac{a'}{\Lambda'}\right)^{n}$$

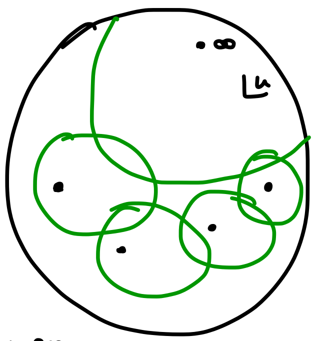
n or 4n?

− or +?

Rotating back to (m,n) $\quad M^{(m,n)} = \begin{pmatrix} 1-mn & n^2 \\ -m^2 & 1-mn \end{pmatrix}$

Picture

$\mathcal{M}_{\text{Coulomb}}$



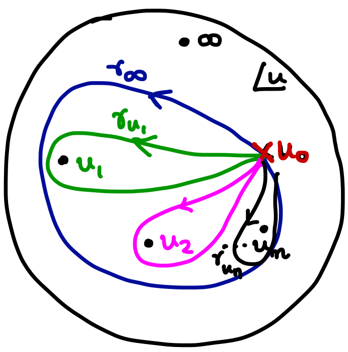
3. In principle what we want to find first of all is how many singularities there are, where they are located in the u-plane, and also what is the massless dyon that is responsible for any singularity (that will tell us about the monodromy matrices) and we'd also like to know the coefficients that appear in the summations for the instanton corrections in the Taylor series.

1. We have a singularity at ∞, and in addition we can have other singularities corresponding to strongly coupled regions.

2. Associated with each of these singularities, there will be some preferred weakly coupled variables (e.g. a, a'). When we expansion express everything in terms of these weakly coupled variables, we have some logarithmic terms which are determined by the 1-loop running and are responsible for monodromies, and in addition there are Taylor series that take into account instanton corrections or corrections due to integrated out massive states.

- Around each singularity there is a patch over which there is a weak-coupling description in the form of an expansion.
- We want to patch together these local descriptions in a globally consistent way, to completely fix the IR effective action.

Global Consistency Condition on the Monodromy Group



To compute the monodromy, we start from a base point u_0, and encircle singularities. The blue loop is to compute the monodromy around ∞. (γ_∞)

The blue loop γ_∞ can be deformed into the union or composition of the loops γ_{u_k} for $k = 1, \ldots, n$.

Blue : Loop around ∞ (γ_∞)
Green : Loop around u_1 (γ_{u_1})
Pink : Loop around u_2 (γ_{u_2})
⋮
Black : Loop around u_n (γ_{u_n})

So the monodromy matrix at ∞ should satisfy

$$\boxed{M_\infty = M_{u_n} \cdots M_{u_2} M_{u_1}}$$

Inherent in the monodromy is a choice of a base point.

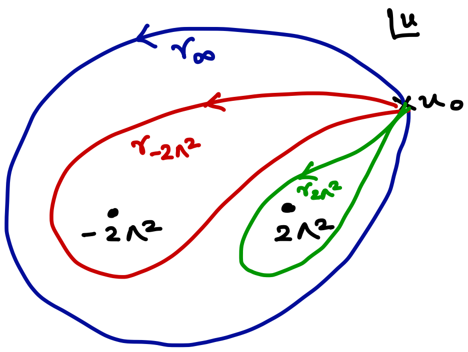
The above picture should be thought of as a sphere with a number of punctures.

Group of loops based at u_0 : $\pi_1(\mathcal{M}_{\text{coulomb}}, u_0)$: Fundamental Group of the Coulomb Branch

Next to Simplest : 2 Singularities at Strong Coupling

There is a $\mathbb{Z}_2$ R-symmetry that is broken on the moduli space.

$$u = 2\Lambda^2 \ , \quad u = -2\Lambda^2$$



Consistency Condition :

$$\boxed{M_\infty = M_{2\Lambda^2} M_{-2\Lambda^2}}$$

These two singularities are interpreted as coming from two massless solitons.

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$$

$M_{\pm 2\Lambda^2}$ of the form $M^{(m,n)}$ for some (m,n)

As we know the form of $M^{(m,n)}$, we can impose the consistency condition to solve for the electric and magnetic charges of the two singularities in moduli space.

Essentially, the only solution (up to conjugation, which does not affect the physics) is as follows:

$$M_{2\Lambda^2} = M^{(1,0)} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = STS^{-1} : \text{ massless monopole}$$

$$M_{-2\Lambda^2} = M^{(1,-2)} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} = T^2 M^{(1,0)} T^{-2} : \text{ massless dyon of mag. charge } 1 \text{ and elec. charge } -2$$

So in particular, the two monodromy matrices associated with the two singularities at strong coupling are related by conjugation, and this is due to the fact that the $\mathbb{Z}_2$ R-symmetry which is broken on the vacuum implies that the physics at $2\Lambda^2$ should be the same as the physics at $-2\Lambda^2$.

__Note__: If we were to start from the base point $-u_0$ (instead of the base point $+u_0$ chosen above), we would have a reversed order of the monodromy matrices in the product, i.e we'd have $M_\infty = M_{-2\Lambda^2} M_{2\Lambda^2}$, and the dyon will have a magnetic charge 1 and an electric charge $+2$.

- By extension of this argument, one might think there could be more than 2 singularities. But this is a well defined mathematics problem and under suitable assumptions it can be shown that the only solution is when there are 2 singularities, with the above monodromy matrix with the singularities being associated with a massless monopole and a massless dyon.

MONOPOLE CONDENSATION & CONFINEMENT IN $\mathcal{N}=1$ SU(2) SYM

Softly breaking $\mathcal{N}=2 \longrightarrow \mathcal{N}=1$. Introduce a superpotential that gives a mass to the scalar ϕ (chiral superfield in the $\mathcal{N}=2$ vector multiplet)

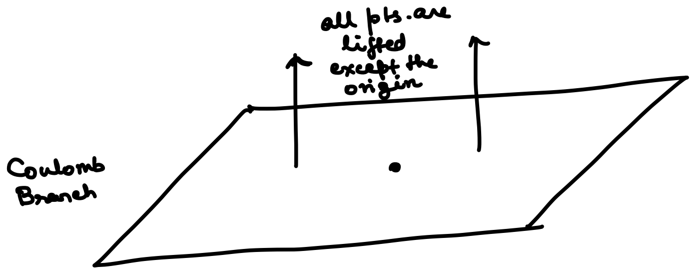
$$\boxed{W = \frac{m}{2} \operatorname{Tr} \phi^2}$$

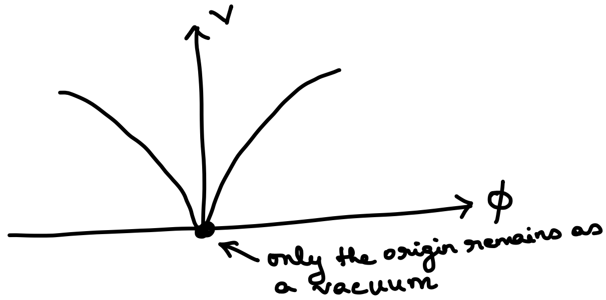
(Integrating out the massive scalar one is just left with the $\mathcal{N}=1$ massless vector multiplet.)

In particular this breaks $SU(2)_R \times U(1)_R$ R-symmetry of the $\mathcal{N}=2$ theory to a single $U(1)_{R'}$ R-symmetry which gives charge $+1$ to ϕ so the superpotential has charge 2. This $U(1)_{R'}$ is broken by an anomaly to $\mathbb{Z}_4$:

$$\underbrace{SU(2)_R \times U(1)_R}_{\text{Classical R-Symmetry of original } \mathcal{N}=2 \text{ theory}} \xrightarrow[\text{breaking}]{\text{soft SUSY}} \underbrace{U(1)_{R'}}_{\substack{\text{gives charge} \\ 1 \text{ to } \phi \text{ and} \\ \text{so superpotential} \\ \text{has charge 2}}} \xrightarrow{\text{anomaly}} \mathbb{Z}_4$$

Classically, the superpotential gives mass to the complex scalar ϕ. So, one expects the Coulomb Branch to get lifted.





Now we would like to study the effect of the soft breaking term in the low energy effective theory taking into account quantum effects.

So at least if the mass is small, one expects that one can take the soft breaking term into account by adding the following term to the effective action

effective superpotential : $W_{eff} = mu$ (recall that $u = \langle \frac{1}{2} \mathrm{Tr}\, \phi^2 \rangle$)

- Seiberg and Witten then argued using holomorphy that this is valid not just for small values of m, but for any value of m.

At a regular point in $\mathcal{M}_{coulomb}$, $F_u \neq 0$ (as $F_u \propto m$)

scalar potential : $V = \underbrace{K^{u\bar{u}}}_{\text{inverse Kähler metric}} F_u F_{\bar{u}} > 0 \rightarrow$ lifted

(at a regular point, the Kähler metric is invertible)

At a singular point, the Kähler metric is not invertible. Let's see what actually happens.

At a singular point, $u = 2\Lambda^2$ (massless monopole) $\rightarrow$ couples linearly to dual special coordinate

$$a_D = c_0 (u - 2\Lambda^2)$$

$\uparrow$ mass of magnetic monopole

$$W_{eff} = mu + c_0 (u - 2\Lambda^2) M\tilde{M}$$

where M & $\tilde{M}$ are the $\mathcal{N}=1$ chiral superfields in the monopole hypermultiplet

We try to solve F-term equations for this superpotential.

$$0 = \frac{\partial W_{eff}}{\partial u} = m + c_0 M\tilde{M}$$

$$0 = \frac{\partial W_{eff}}{\partial M} = c_0 (u - 2\Lambda^2) \tilde{M}$$

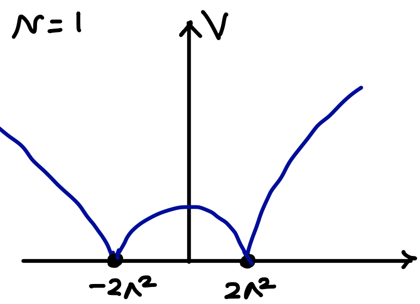
$$0 = \frac{\partial W_{eff}}{\partial \tilde{M}} = c_0 (u - 2\Lambda^2) M$$

$\rightarrow$ There is a simple solution $\boxed{u = 2\Lambda^2}$ (pt. where the monopole is massless) which solves the last 2 equations. Then the first one is solved by

$$\boxed{M\tilde{M} = -\frac{m}{c_0}}$$

What we've found is that the point $u = 2\Lambda^2$ on the Coulomb Branch corresponding to the singularity in the $\mathcal{N}=2$ moduli space where the monopole is massless survives when we softly break $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ supersymmetry.

- One can repeat the same analysis for the dyon, replacing the monopole hypermultiplet with the dyon hypermultiplet.



Two supersymmetric vacua.
At a regular point in the moduli space, SUSY is broken and the potential is positive. But at the two singular points, we still have SUSY vacua.

▶ The vacuum corresponding to the massless monopole is given by the solution to the F-term equations (found above)

$$\boxed{\begin{aligned} &(\mathrm{I}) \quad u = 2\Lambda^2 \\ &(\mathrm{II}) \quad M\tilde{M} = -\frac{m}{c_0} \end{aligned}}$$

Monopole Condensation

Not only must the scalar field in the $\mathcal{N}=2$ vector multiplet (which is massive) take the value corresponding to the singular point (I), but also in addition, the magnetic monopole has to acquire an expectation value (II). This is called MONOPOLE CONDENSATION.

▶ By a similar argument, at $u = -2\Lambda^2$, one has DYON CONDENSATION.

As monopoles are magnetically charged, when they condense, they give rise to a Meissner Effect. MAGNETIC HIGGS MECHANISM. This is the magnetic version of the Meissner Effect.

Confinement of electric charge occurs due to this "Meissner Effect".

(confinement for electric charge)

We go back to SU(2) $\mathcal{N}=2$ Super Yang Mills Theory now.

PHYSICS PROBLEM : Determine IR EA of $\mathcal{N}=2$ pure SU(2) YM

$\downarrow$

MATHEMATICS PROBLEM : Finding multivalued functions $\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$ of u (u is the coordinate parametrizing the Coulomb Branch) mathematically, holomorphic $SL(2,\mathbb{Z})$ sections (or special coordinates) with prescribed monodromies $M_{\infty, \pm 2\Lambda^2}$ at 'singularities' $\infty, \pm 2\Lambda^2$, such that

$$\tau(u) = \frac{\partial a_D}{\partial a}$$

has positive imaginary part (due to unitarity).

This mathematics problem is in fact well-studied and falls into the class of Riemann–Hilbert problems. It is known that it has a unique solution up to multiplication of $a(u)$ and $a_D(u)$ by an entire function.

Multiplying a and a_D by the same entire function will not change the monodromy. We can physically fix this ambiguity by imposing the correct weak coupling asymptotics.

Monodromy Group : generated by any two of the 3 monodromy matrices

$$\Gamma_0(4) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \beta = 0 \bmod 4 \right\} \longleftarrow \text{Monodromy Group}$$

Note that the S-transformation does not belong to the monodromy group:

$$S \notin \Gamma_0(4)$$

The T-transformation also does not belong to the monodromy group:

$$T \notin \Gamma_0(4)$$

However, $T^4 \in \Gamma_0(4)$.

As u varies along $\mathcal{M}_{\text{coulomb}}$, $\tau(u)$ varies in the UHP

$$\mathcal{H}^+ = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}.$$

Monodromies around singularities in $\Gamma_0(4)$.

but is also subject to monodromies as we go around singularities in the u plane.

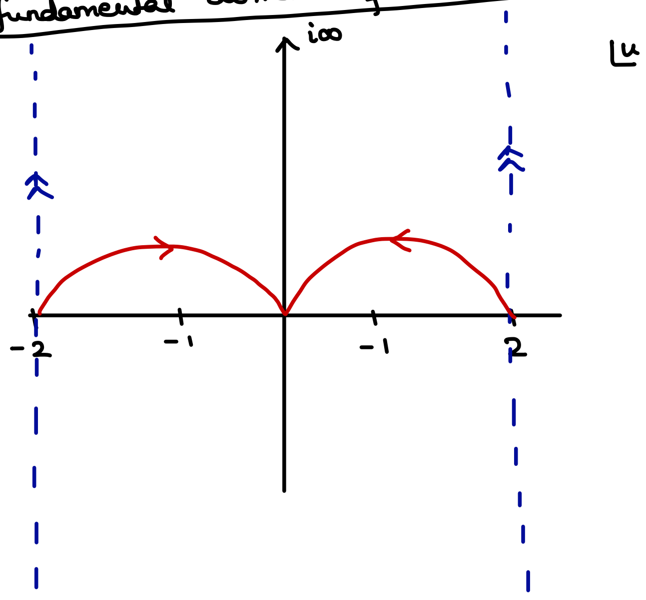
So one should think of the upper half plane $\mathcal{H}^+$ as a cover of the u-plane, where the cover is due to monodromies that get picked up around singularities.

$$\boxed{\mathcal{M}_{\text{coulomb}} \cong \mathcal{H}^+/\Gamma_0(4)}$$

$\mathcal{M}_{\text{coulomb}}$: parametrized by u

$\mathcal{H}^+/\Gamma_0(4)$: parametrized by $\tau(u)$ up to the action of the monodromy group $\tau(u) \sim M \cdot \tau(u)$

The fundamental domain of the monodromy group $\Gamma_0(4)$



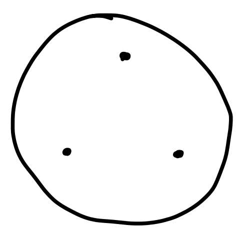
① Can use T-transformation to restrict to $\text{Re}(u) \in [-2,2]$

② $\hat{\hat{}}$ $\Rightarrow$ identify edges

③ can use $M_{-2\Lambda^2}$ or $M_{2\Lambda^2}$

④ $\rightarrow$ $\leftarrow$ $\Rightarrow$ identify

Topologically



Sphere with 3 punctures.
The 3 punctures correspond to the 3 cusps.
The cusps are $\tau = i\infty$
$\tau = 0$
and $\tau = 2$ (eq. to -2)

EXERCISE: (1) Motivate the above identifications.

(2) Identify cusps as fixed points of elements of $\Gamma_0(4)$

cusp at $\tau = 0 \iff$ magnetic monopole $(1,0)$ $\longrightarrow u = 2\Lambda^2$

cusp at $\tau = 2 \iff$ massless dyon $(1,-2)$ $\longrightarrow u = -2\Lambda^2$

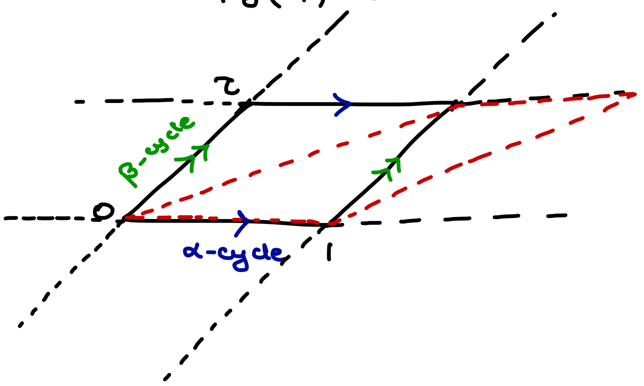
cusp at $\tau = -2 \iff$ massless dyon $(1,2)$ $\longrightarrow u = -2\Lambda^2$

cusp at $\tau = i\infty \iff$ $u = \infty$
weak coupling point

▶ The Coulomb Branch is isomorphic to the fundamental domain of the monodromy group. The mondromy group itself is a subgroup of $SL(2,\mathbb{Z})$.

▶ $SL(2,\mathbb{Z})$ is the group of large diffeomorphisms of the 2-torus (T^2). This means the group of diffeomorphisms modulo the diffeomorphisms that are continuously connected to the identity.

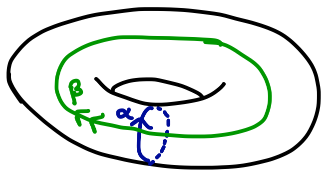
$$\Gamma_0(4) \in SL(2,\mathbb{Z}) \longleftarrow \text{Large diffeos of } T^2 \cong \mathbb{C}^2/\mathbb{Z}+\tau\mathbb{Z}$$



τ : complex structure of the 2-torus (called the period matrix, in the context of Riemann surfaces)

$$T : \tau \longrightarrow \tau + 1 \; ; \; \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \beta+\alpha \\ \alpha \end{pmatrix}$$

taking red dotted part as the new fundamental domain



$$S : \tau \longrightarrow -\frac{1}{\tau} \; ; \; \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \xrightarrow{S} \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}$$

Consider any loop or 1-cycle in the 2-torus.

1-cycle : $\gamma \in H_1(T^2, \mathbb{Z})$ has winding number (m,n) : expresses how γ is composed in terms of the basis of 1-cycles given by α & β.

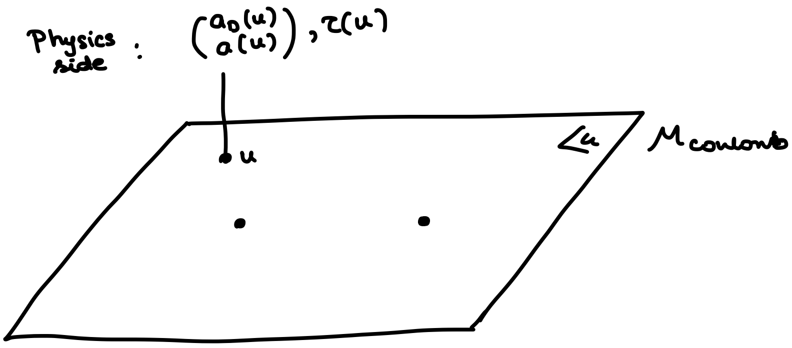
$$\gamma = m\beta + n\alpha = (m,n)\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

We encounter a structure similar to what we had before.

MATH SIDE			PHYSICS SIDE	
Homology Cycles	$\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$	$\longleftrightarrow$	$\begin{pmatrix} a_D \\ a \end{pmatrix}$	special coordinates
Winding numbers	(m,n)	$\longleftrightarrow$	(m,n)	electric & magnetic charges
Complex structure modulus	τ	$\longleftrightarrow$	τ	complexified gauge coupling
Large diffeomorphisms	$SL(2,\mathbb{Z})$	$\longleftrightarrow$	$SL(2,\mathbb{Z})$	electric-magnetic duality group

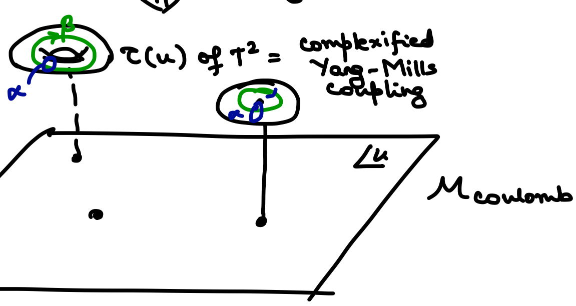
As the structures on the physics and math side are very similar, what Seiberg and Witten did was to identify the two sides:

Homology cycles	$\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$	$\xleftrightarrow[\text{made more precise}]{\text{to be}}$	$\begin{pmatrix}a_D\\ a\end{pmatrix}$	special coordinates
Winding numbers	(m,n)	$=$	(m,n)	electric & magnetic charges
Complex structure modulus	τ	$=$	τ	complexified gauge coupling
Large diffeomorphisms	$SL(2,\mathbb{Z})$	$=$	$SL(2,\mathbb{Z})$	electric-magnetic duality group



As τ is the complex structure of a 2-torus, what Seiberg and Witten said was that we encode all the information on the physics side (the special coordinates and τ) in terms of an auxiliary 2-torus which is fibred over the complex plane.

$\Downarrow$ Seiberg & Witten



1-parameter family of genus-1 Riemann surfaces (isomorphic to the 2-torus) varies holomorphically with u $\quad X_u$

(holomorphicity is due to $\mathcal{N}=2$ SUSY).

We have a bundle where the base is the Coulomb Branch and the fiber is given by the 2-torus, whose complex structure determines the value of the coupling. At the singular point, one of the cycles of the 2-torus will vanish. (In the diag. shown, (contd.)

[At the singular point, one of the cycles of the β-cycle vanishes).]

We also want the complex structure moduli space to be $\mathcal{H}^+/\Gamma_0(4) \equiv \mathcal{M}_{coulomb}$.

X_u The 2-torus is described by an algebraic equation of the form

$$\boxed{y^2 = (x^2-u)^2 - 4\Lambda^4}$$

"SEIBERG-WITTEN CURVE"

1-equation in x & $y \in \mathbb{C}$
u is a parameter.

$$\Downarrow$$

$$\boxed{y^2 = \prod_{i=1}^{4}(x - e_i(u,\Lambda))}$$

$\rightarrow$ determines y as a double cover of the x-plane (double cover due to the square root)

The four roots are $\quad e_1 = -\sqrt{u+2\Lambda^2}\,,\quad e_2 = -\sqrt{u-2\Lambda^2}$

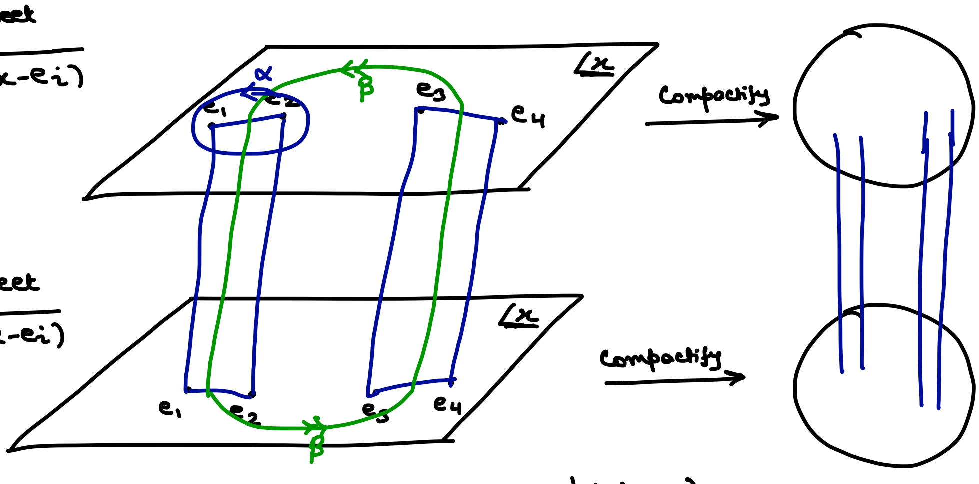
$$e_3 = +\sqrt{u-2\Lambda^2}\,,\quad e_4 = +\sqrt{u+2\Lambda^2}$$

Curve determines y as a double cover of the x-plane, branched at e_1, e_2, e_3, e_4.

Riemann-sphere (as x is complex)

Upper Sheet
$y = \sqrt{\prod_i (x - e_i)}$

Lower Sheet
$y = -\sqrt{\prod_i (x - e_i)}$



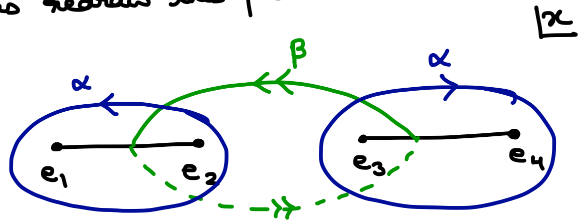
Riemann sphere

= Torus
(the identifications introduce handles)

Riemann sphere

(I think Penrose's book has a nice picture)

Let us redraw this picture:



As the x-plane is now a Riemann sphere, the cycles can be deformed

$\{\alpha, \beta\}$: canonical homology basis for $H_1(X, \mathbb{Z})$

INTERSECTION PAIRING :
$$\alpha \circ \alpha = 0$$
$$\beta \circ \beta = 0$$
$$\alpha \circ \beta = -\beta \circ \alpha = 1$$

LOW ENERGY COUPLING $\tau(u)$ = COMPLEX STRUCTURE MODULUS (PERIOD MATRIX) OF X_u.

$$\boxed{\omega = \frac{dx}{y(x)}}$$

HOLOMORPHIC DIFFERENTIAL (Holomorphic (1,0) form) → The space of holomorphic differentials on a genus-g Riemann surface has dimension g. So for a T^2 ($g=1$), there is a unique holomorphic differential (unique up to multiplication by a constant).

Holomorphic on the Riemann sphere (closure of the complex x plane)

[EXERCISE: Check that w is indeed holomorphic, i.e. is regular on the Riemann sphere for x.]

- One might worry that $x = e_i$ and $x = \infty$ are dangerous points. But plugging in these x values, suitably changing variables reveals that w is indeed regular.

$$\boxed{\tau(u) = \frac{\oint_\beta \omega}{\oint_\alpha \omega}}$$

Complex Structure of the T^2

"periods" of w along β & α cycles

Riemann's 2^{nd} Bilinear Identity : $\mathrm{Im}(\tau) > 0$ (τ given by the above formula)

For us this is crucial as this is the condition given by unitarity.

[Note: $\mathrm{Im}(\tau) > 0$ for a regular torus. If the torus develops a singularity, the imaginary part of τ can go to zero.]

This was the geometric side.

- From the physics side,

$$\boxed{\tau(u) = \frac{\partial a_D(u)}{\partial a(u)} = \frac{\partial a_D(u)/\partial u}{\partial a(u)/\partial u}}$$

(u is the global coordinate on the Coulomb Branch)

Equating the physics and math side expressions, we can infer for τ

$$\frac{\partial a_D}{\partial u} = \cancel{f(u)} \oint_\beta \omega$$

$$\frac{\partial a}{\partial u} = \cancel{f(u)} \oint_\alpha \omega$$

- We don't want a fxn $f(u)$ to premultiply as it will in general introduce some extra poles and zeroes.
- $f(u) \longrightarrow$ constant, which can be fixed by normalizing the integral of the holomorphic differential.

$$\boxed{\begin{aligned} \frac{\partial a_D}{\partial u} &= c \oint_\beta \omega \\ \frac{\partial a}{\partial u} &= c \oint_\alpha \omega \end{aligned}}$$

c = constant

But to compute central charges or masses of BPS objects, we want a formula expressing a_D and a, as fxns of u.

So we integrate these equations, by introducing another differential λ such that

$$\boxed{\frac{\partial \lambda}{\partial u} = \omega}$$

$$\boxed{\begin{aligned} a_D(u) &= \oint_\beta \lambda \\ a(u) &= \oint_\alpha \lambda \end{aligned}}$$

λ is called the SEIBERG-WITTEN DIFFERENTIAL.

Unlike ω which was holomorphic, λ is MEROMORPHIC.

So in particular, λ has some poles, but no residues (?) Mathematically, it is an abelian differential of the second kind.

Also, λ is defined up to an exact form :

$$\lambda \sim \lambda + d(\ldots)$$

as we only care about its periods (i.e. loop integrals of λ).

In this case, the meromorphic differential which does the job in the case of $SU(2)$ using the parametrization $y^2 = (x^2-u)^2 - 4\Lambda^4$, is

$$\boxed{\lambda = \frac{1}{2\pi i} \frac{2x^2\, dx}{y(x)}}$$

There are many equivalent descriptions that differ by exact pieces or changes of variables

another form

$$\boxed{\lambda = \frac{1}{2\pi i} dx \, \mathrm{arccosh}\left(\frac{x^2-u}{2\Lambda^2}\right) + d(\cdot\cdot)}$$

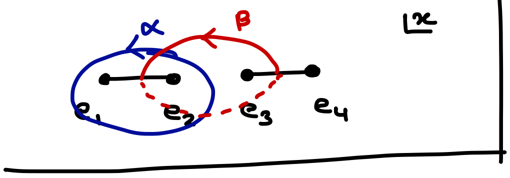
- λ is a meromorphic differential. In particular, it has a double pole at $x=\infty$ (with no residue).

NEED TO CHECK

(1) X_u degenerates at $u = \pm 2\Lambda^2, \infty$

(2) $\begin{pmatrix} a_D \\ a \end{pmatrix}$ have the correct asymptotics at singularities $\longrightarrow$ correct monodromies

SINGULARITIES OF X_u



If the branch points are all distinct, the torus is regular. On the other hand, the torus will become singular when 2 branch points collide.

Branch points collide $\longleftrightarrow$ 1-cycle vanishes.

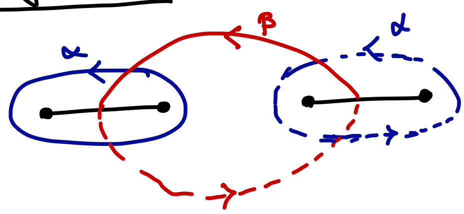
$\Longleftrightarrow$ vanishing of the discriminant of $y^2 = (x^2-u)^2 - 4\Lambda^4$

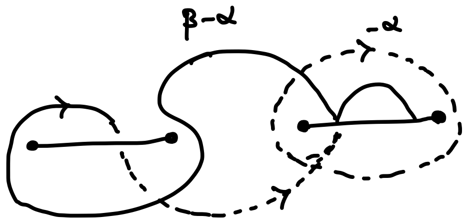
$$\Delta(u) = \prod_{i<j} (e_i - e_j)^2 \propto \Lambda^d (u^2 - 4\Lambda^4)$$

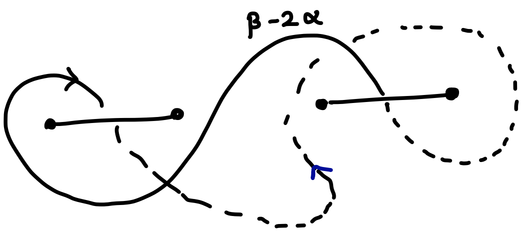
Zeroes of the discriminant : $u^2 = 4\Lambda^4 \Rightarrow u = \pm 2\Lambda^2$

- $u \longrightarrow 2\Lambda^2$; $e_2 \longrightarrow e_3$, $\nu_{2\Lambda^2} = \beta$ vanishes $\Rightarrow \tau \to 0$
- $u \longrightarrow -2\Lambda^2$; $e_1 \longrightarrow e_4$, $\nu_{-2\Lambda^2} = \beta - 2\alpha$ vanishes $\Rightarrow \tau \to 2$

Why $(\beta - 2\alpha)$?







$\longleftarrow$ understand more carefully

- $u \longrightarrow \infty$ extra singularity; the discriminant does not vanish but some dimensionless version goes to zero, and $\tau \longrightarrow i\infty$. There is no cycle which vanishes.

  e_1 & e_2 are comparable

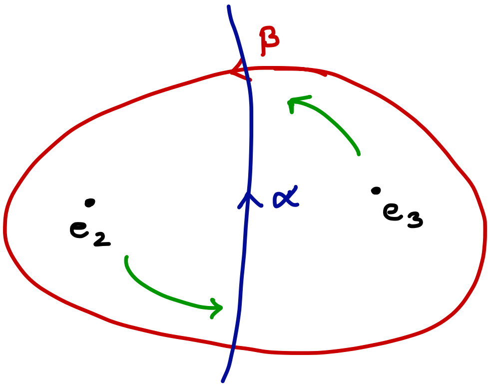
MONODROMY AT $u = 2\Lambda^2$: $u - 2\Lambda^2 \longrightarrow e^{2\pi i}(u-2\Lambda^2)$

This operation exchanges e_2 and e_3. e_1 & e_4 remain spectators. Under this operation,

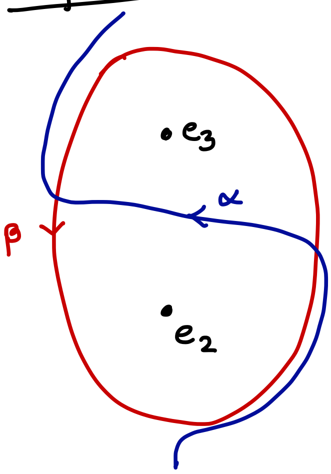
$$\beta \longrightarrow \beta \qquad \text{vanishing cycle remains the same}$$
$$\alpha \longrightarrow \alpha - \beta$$

We only have to focus on e_2 and e_3

Green arrows represent the effect of the operation $u - 2\Lambda^2 \longrightarrow e^{2\pi i}(u-2\Lambda^2)$.
This is visualized in two steps, i.e. through half-rotations

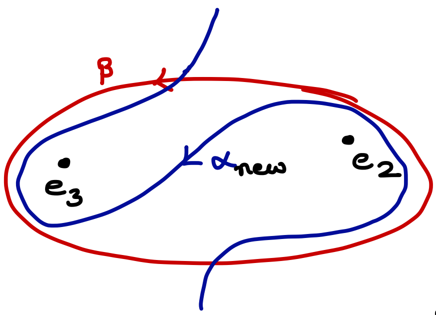


Half rotation #1



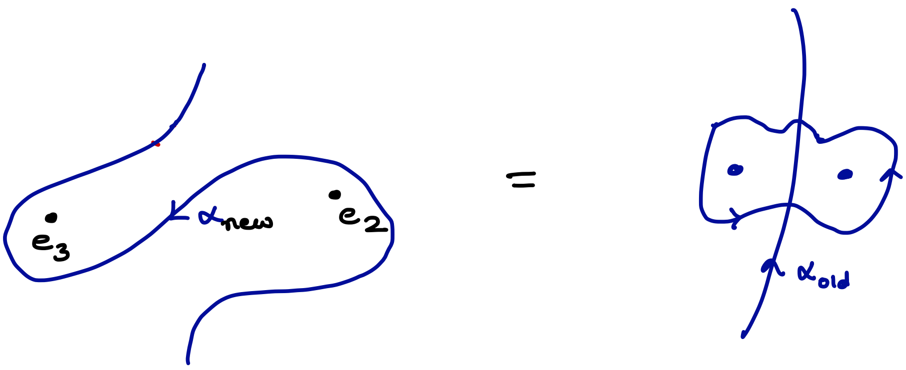
$\rightsquigarrow$

Half rotation #2



This is the final picture under monodromy at $u = 2\Lambda^2$.

The relation between α_{new} and α_{old} is clearer on drawing the final picture without β first and comparing it to the original picture prior to the monodromy operation.



This picture isn't very clear.

Therefore the argument for $\alpha - \beta$ isn't v. clear either.

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$$

monodromy matrix

This is just $M^{(1,0)}$ (magnetic monopole)

$\therefore$ $\boxed{M_{2\Lambda^2} = M^{(1,0)}}$ MONODROMY AT $u = 2\Lambda^2$ = Monodromy for a magnetic monopole $(1,0)$

- We can repeat this exercise at the other singularity.

MONODROMY AT $U = -2\Lambda^2$ $(e_1 \to e_4)$

Starting from the results for monodromy at $u = +2\Lambda^2$ (obtained above), one can get the results for monodromy at $u = -2\Lambda^2$ by replacing

$$\beta \to \beta - 2\alpha$$
$$\alpha \to \alpha$$

$$\boxed{M_{-2\Lambda^2} = \begin{pmatrix}-1 & 4\\ -1 & 3\end{pmatrix} = M^{(1,-2)}}$$

MONODROMY AT $u = -2\Lambda^2$ = monodromy for a dyon $(1,-2)$

- In general, we see the following correspondence

SINGULARITY $\longleftrightarrow$ [MASSLESS HYPER charge (m,n) $\updownarrow$ VANISHING CYCLE winding (m,n)] $\longleftrightarrow$ MONODROMY

more precise

$\updownarrow$: Central charge formula & $\begin{pmatrix}a_D\\ a\end{pmatrix} = \oint_{\binom{\beta}{\alpha}} \lambda$

Let $v = m\beta + n\alpha = (m,n)\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$

Then $Z_{(m,n)} = m a_D + n a = (m,n)\begin{pmatrix}a_D\\ a\end{pmatrix} = (m,n)\oint_{\binom{\beta}{\alpha}} \lambda = \oint_{v = m\beta + n\alpha} \lambda$

The central charge $Z_{(m,n)}$ of a dyon of magnetic charge m and electric charge n is nothing but the period of the Seiberg–Witten differential λ. So whenever the cycle with winding number (m,n) vanishes, the dyon with charges (m,n) becomes massless.

Here we have used a specific basis of homology cycles, α and β. One may use a different basis which is related to α and β by symplectic transformation which is $SL(2,\mathbb{Z})$ transformation. That translates on the physics side to a change of electric-magnetic duality frame.

Invariant :

$$v_1 \circ v_2 = (m_1 \beta + n_1 \alpha) \circ (m_2 \beta + n_2 \alpha)$$

intersection number

$$= -m_1 n_2 + n_1 m_2$$

DIRAC PAIRING

PICARD-LEFSCHETZ FORMULA : lets us compute the monodromy M_v matrix associated to a vanishing cycle v.

intersection

$$M_v : \gamma \longrightarrow M_v \gamma = \gamma - (\gamma \circ v) v$$

$$\text{If} \quad v = (m, n) \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \quad \text{then} \quad M_v = M^{(m,n)}$$

- For the pure $SU(2)$, the period integrals of the meromorphic form λ can be computed explicitly and are given in terms of elliptic integrals and can also be written in terms of hypergeometric functions.
- Can get a_D & a as fns of u, expand them in u
  - asymptotic, large u : weak coupling
  - dyon or monopole singularity