HITCHIN SYSTEMS IN SUPERSYMMETRIC FIELD THEORY

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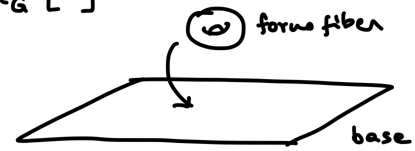
Fix compact Lie group G
Fix a compact Riemann surface C $\Big\} \rightsquigarrow \mathcal{M}_G[C]$

HITCHIN'S INTEGRABLE SYSTEM

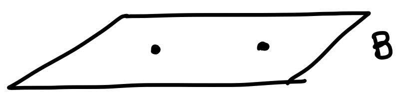


hol. symplectic mfd.
in fact hyperkähler
family of hol. symp. structures labeled $\zeta \in \mathbb{C}P^1$

main point: ① $\mathcal{M}_G[C]$ is part of a SUSY QFT $\mathfrak{X}_g[C \times S^1]$.

② can use this to see new things about $\mathcal{M}_G[C]$.
e.g. distinguished local coordinate systems $\{\mathcal{X}_\gamma(\zeta)\}$, $\zeta \in \mathbb{C}^*$.

$\mathcal{B}$ disc : discriminant locus
$\mathcal{B}_{reg} := \mathcal{B} \setminus \mathcal{B}$ disc



Local system of lattices $\Gamma \longrightarrow \mathcal{B}_{reg}$.
Local coordinate fxns and a part of $\mathcal{M}$ indexed by $\gamma \in \Gamma$.

one use of these coordinates is to construct a hyperKähler metric on this space.

Theory $\mathfrak{X}$

Fix $\mathfrak{g}$ of ADE type.
$\exists$ a 6-dim^l interacting QFT $\mathfrak{X}_g$.
Its believed that $\nexists$ a QFT above $d=6$.)
(and $\nexists$ interacting QFT above $d=6$.)

WHAT ARE THE BASIC FACTS ABOUT $\mathfrak{X}$?

① It has $\mathcal{N}=(2,0)$ susy in 6d. (i.e. it is a superextension of $ISO(5,1)$ which acts on anything in the theory)
$\mathbb{R}^{5,1}$

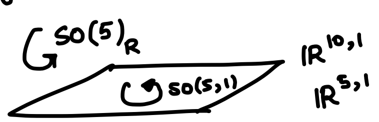
② It has no parameters, i.e. its is totally rigid.

③ $\mathfrak{X}_g$ is conformally invariant.

④ $\mathfrak{X}_g$ is not believed to be given by path integrals over a space of fields with a Lagrangian.

All known constructions today use string theory.

⑤ $\mathfrak{X}_g$ has local operators ϕ^i_F labeled by • $i = 1, \ldots, 5$ $\quad 5 = 11-6$
• F invariant polynomial in $\mathfrak{g}$.

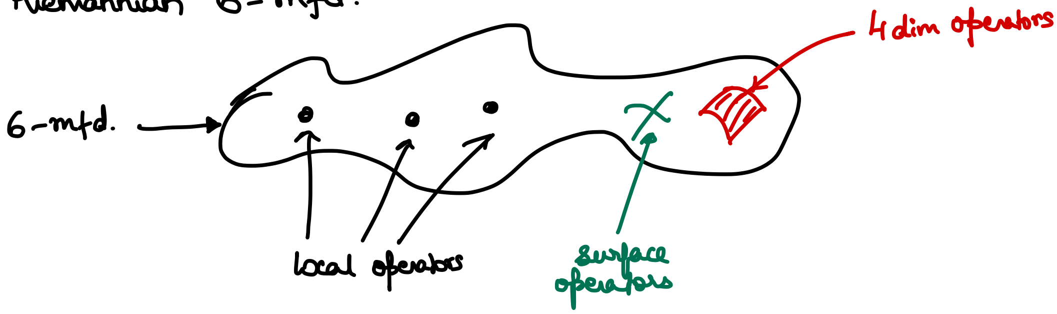


example • if $\mathfrak{g} = A_{K-1}$, then $\phi^i_2, \ldots, \phi^i_K$ corresponding to invariant polynomials
$F(X) = \mathrm{Tr}(X^n) \quad n = 2, \ldots, K.$

⑥ $\mathfrak{X}_g$ has surface operators labeled by irreps of $\mathfrak{g}$ plus a vector in $\mathbb{R}^5$

⑦ $\mathfrak{X}_g$ has 4-dimensional operators labeled by (roughly) nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ plus a 2-plane in $\mathbb{R}^5$

So roughly $\mathcal{X}_g$ (Euclidean version) should assign a number to a compact Riemannian 6-mfd.



WHAT IS COMPACTIFICATION?

Given a QFT T in d-dim

instead of taking spacetime to be $\mathbb{R}^{d-1,1}$

take it to be $Y \times \mathbb{R}^{d-k-1,1}$, Y : Riemannian k-manifold

(This may involve some choices!)

Get a theory $T[Y]$ with $ISO(d-k-1,1)$

lower bosonic symmetry

maybe some SUSY (if T had SUSY in the first place)

- want to do this for $\mathcal{X}_g$.
- It is possible to compactify $\mathcal{X}_g$ on S^1, of length $R \rightsquigarrow T(S^1)$.

"If we consider $T[S^1]$ at energy scales $\ll \frac{1}{R}$, then the physics is that of 5-dimensional supersymmetric Yang-Mills theory with gauge group G, and coupling $g_{YM}^2 \sim R$."

WHAT IS 5d SYM?

Fields:
- G-bundle P with connection D
- sections Φ^i of $Ad(P)$, $i = 1, \dots, 5$
- fermions

Lagrangian:

$$\mathcal{L} = \frac{1}{8\pi^2 R} \operatorname{Tr}\Big[F_D \wedge * F_D + R^2 \sum_{i=1}^{5} D\Phi^i \wedge * D\Phi^i + \text{fermions}\Big].$$

interacting! (if $G \neq$ Abelian)

$(D = d + A)$

$$F = dA + A \wedge A$$

(would've been free if G = Abelian)

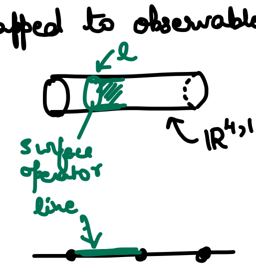
[Ad P is the trivial bundle if G = Abelian]

Observables of $\mathcal{X}_g[S^1]$ can be mapped to observables of 5d SYM

e.g. $\phi_F^i(\underset{\mathbb{R}^{5,1}}{\vec{x}}, \underset{S^1}{x^6})$

$\Big\downarrow$

$F(\Phi^i(\vec{x}))$



surface defect in reg^n R

$\Big\downarrow$

(SUSY) Wilson line

Tr Holonomy $(D + \Phi\, ds)$

Saying that $\mathfrak{X}_g[S^1]$ reduces to 5d SYM at $E << \frac{1}{R}$ means e.g. that if we compute

the Fourier transform of $\left\langle \prod_i \mathcal{O}_{UV}(\vec{x}_i, x_i^6) \right\rangle$ in $\vec{X}$ directions

and compare it to

the Fourier transform of $\left\langle \prod_i \mathcal{O}_{IR}(\vec{x}_i, x_i^6) \right\rangle$ "

they agree up to modes of order $\|\vec{k}\| R$.

▶ We should <u>not</u> think of passing to the IR as a poor man's substitute for having the whole UV theory.

Lecture 2

Last time, $\mathfrak{X}_g[S^1]$ looks in the low-energy limit $(E << \frac{1}{R})$ like 5d SYM.

C is a Riemannian 2-manifold

It is possible to compactify $\mathfrak{X}_g$ on C in such a way that $\mathfrak{X}_g[C]$ has $ISO(3,1)$ invariance and $\mathcal{N}=2$ supersymmetry.

In the limit where we rescale the metric of C

$$g_C \longrightarrow t\, g_C \qquad , \quad t \to 0$$

the limit

$$\lim_{t \to 0} \mathfrak{X}_g[C, tg]$$

<u>exists</u>, depends only on the <u>conformal class</u> of metric on C (i.e. C as a Riemann surface)

▶ " Theory of Class S "

■ Observables on $\mathfrak{X}_g$ map to observables on $\mathfrak{X}_g[C]$.

e.g. – Local operators on $\mathfrak{X}_g$ map to local operators on $\mathfrak{X}_g[C]$

× $\mathbb{R}^{3,1}$

– Surface operators on $\mathfrak{X}_g$ map to

pt on C

× surface $\subset \mathbb{R}^{3,1}$

• surface operators on $\mathfrak{X}_g[C]$

• line operator on $\mathfrak{X}_g[C]$

loop on C

× line in $\mathbb{R}^{3,1}$

4d operators could lead to domain walls e.g.

• local operator on $\mathfrak{X}_g[C]$

▶ FOR GENERAL CHOICES OF g and C, $\mathfrak{X}_g[C]$ IS NOT KNOWN TO BE LAGRANGIAN. Sometimes it is.

- We can also consider $\mathcal{X}_g[C \times S^1]$. Get a theory with $\mathcal{N}=4$ SUSY in 3d.

Two different perspectives
- size $(S^1) \ll$ size (C)
- size $(S^1) \gg$ size (C).

Remarkably, it seems that both perspectives work outside these naive domains!

$\mathcal{X}_g[C \times S^1]$ with size $(S^1) \ll$ size (C)

In this perspective, what we have is really 5d SYM compactified on C.

Fields of 5d SYM : $A \quad , (\mathcal{D} = d + A)$, $\underbrace{\Phi^1, \Phi^2, \Phi^3, \Phi^4, \Phi^5}$ } flat space

combine them into a complex valued 1-form on C
$$\varphi = \Phi^4 + i\Phi^5$$

so on C, the fields are
$$A, \underbrace{\Phi^1, \Phi^2, \Phi^3}_{\text{3 real scalars}}, \varphi$$
φ ← complex-valued 1-form on C

How to describe the physics at low energies $\left(E \ll \frac{1}{|C|}\right)$?

- 5d SYM is IR free $\Rightarrow$ can use classical field theory to answer this question.

- What are the classical field configurations which preserve $ISO(2,1)$ symmetry and the supersymmetry?

<u>Answer</u> : assuming $\Phi^1 = 0$, $\Phi^1 = \Phi^2 = \Phi^3 = 0$,

then
$$\boxed{\begin{aligned} F_{\mathcal{D}} + R^2[\varphi, \varphi^\dagger] &= 0 \\ \bar{\partial}_{\mathcal{D}} \varphi &= 0 \end{aligned}}$$
— (*) HITCHIN'S EQUATIONS

the equation $\bar{\partial}_{\mathcal{D}} \varphi = 0 \to \varphi$ is covariantly holomorphic

(*) are Hitchin's equations

$$\mathcal{M}_C[G] = \{\text{solutions of } *\} / \{\text{gauge transformations}\}$$

is Hitchin's integrable system.

- moduli space of $\mathcal{X}_g[C \times S^1]$.

If we compute <u>correlation functions</u> of $\mathcal{X}_g[C \times S^1]$ on $\mathbb{R}^{2,1}$?

What we get is a function on $\mathcal{M}_C[G]$.

[<u>Why?</u> e.g. to formulate the path integral we need to fix boundary condition for the fields $(\mathcal{D}, \varphi) \longrightarrow (\mathcal{D}_0, \varphi_0)$ on $\mathcal{M}_C[G]$.]

Remark: the supersymmetry also imposes equations on Φ^1, Φ^2, Φ^3:

$$D\Phi^i = 0 \quad , \quad [\varphi, \Phi^i] = 0 \qquad (i = 1,2,3)$$
$$[\Phi^i, \Phi^j] = 0$$

(φ is called the "Higgs field")

What we defined above is a branch of the moduli space, called the Hitchin moduli space.

$\mathcal{M}_C[G]$
Coulomb branch

mixed / Higgs branches

Note: what mathematicians call the moduli of Higgs bundles really live on the Coulomb branch!

The low-energy physics of $\mathfrak{X}[C \times S^1]$

$\parallel$

is a "σ-model with target $\mathcal{M}_C[G]$"

- field theory with fields : * maps $\sigma : \mathbb{R}^{2,1} \longrightarrow \mathcal{M}_C[G]$
  * fermions
- $\mathcal{N} = 4$ SUSY $\Rightarrow$ metric on $\mathcal{M}_C[G]$ is hyperKähler.

We would like to get some new information about $\mathcal{M}_C[G]$ from this point of view.

- We want to understand $\mathcal{M}_G[C]$ better by using $\mathfrak{X}_{\mathfrak{g}}[C \times S^1]$ as a compactification of $\mathfrak{X}_{\mathfrak{g}}[C, 0]$ on S^1 (now we're in the regime $|C| << |S^1|$)
- first have to understand the low-energy physics of the 4d theory $\mathfrak{X}_{\mathfrak{g}}[C, 0]$ itself.

Like $\mathfrak{X}_{\mathfrak{g}}[C \times S^1]$ (the 3d theory), the 4d theory $\mathfrak{X}_{\mathfrak{g}}[C, 0]$ has a moduli space of vacua.

- Coulomb branch $\mathcal{B}$: Hitchin base of $(\mathfrak{g}, C)$

  e.g. if $\mathfrak{g} = A_{K-1}$, $\quad \mathcal{B} = \bigoplus_{r=2}^{K} H^0(C, K_C^{\otimes r})$

  complex vector space

Coulomb branch	complex mfd. B	Hitchin base
singular locus	singular $\mathcal{B}_{sing} \subset \mathcal{B}$ (distinguished divisor)	discriminant locus of the Hitchin base
EM charge lattices	local system of lattices $\Gamma \longrightarrow \mathcal{B} \setminus \mathcal{B}_{sing}$	$H_1(\Sigma_u, \mathbb{Z})$
central charges	$Z : \Gamma \longrightarrow \mathbb{C}$ fiberwise homomorphism varies holomorphically over the Coulomb branch	$Z_\gamma = \oint_\gamma \lambda$

The charge lattice comes with an antisymmetric pairing.

$$\langle , \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$$

The constraint is

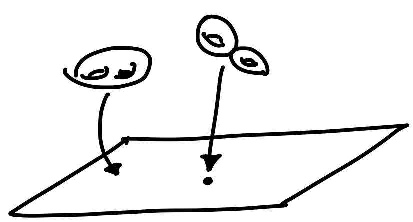
$$\langle dZ, dZ \rangle = 0$$

(this corresponds to the existence of a prepotential from the perspective of $\mathcal{N}=2$ SUSY.)

$$\Sigma_u = \text{spectral curve} \subset T^*C$$

e.g. at $(\phi_2, \dots, \phi_K) \in \mathcal{B}$

$$\Sigma_u = \left\{ \lambda^K + \sum_{r=2}^{K} \phi_r \lambda^{K-r} = 0 \right\}$$



We have a cpx. mfd. with a local system of lattices over it, which are just the homology of a family of curves

at some pts the curve degenerates (at the singular points)

in order to understand the low-energy physics of this theory, all we need is this family of curves; more precisely we need the family of their homologies and the central charge functions.

- **Key statement** [Seiberg – Witten]

If $u \in B \setminus B_{sing}$, the low-energy physics (lower than $\min_{\gamma \in \Gamma} |Z_\gamma|$) is given by 4d SYM with abelian gauge group $G = U(1)^r$ ($r = \dim_{\mathbb{R}} \mathcal{B}$).

▸ At the special points, one of the cycles of the spectral curves shrinks. This corresponds to one of the Z_γ's going to zero.

- We want to next study the compactification of the Hitchin Base — with the above interpretation — on a circle.