Lectures on Supersymmetric Path Integrals

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Part I : SUSY on Curved Spaces

contents

Spinors in 3D

- Gamma matrices for $SO(1,2)$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

- generators of Lorentz transformation in <u>spinor rep^n</u> : $\frac{1}{2}\gamma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$
- Lorentz transformation on vectors and spinors :

$$\delta v^a = \omega^a{}_b v^b$$
$$\delta \psi^\alpha = \frac{1}{4}\omega_{ab}(\gamma^{ab})^\alpha{}_\beta \psi^\beta$$

- $\{\gamma^{01}, \gamma^{02}, \gamma^{03}\}$ span the linear space of 2×2 real traceless matrices, so

$$SO(1,2) \cong SL(2,\mathbb{R})$$

- invariant inner products of spinors

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta, \quad C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left[\text{PvN}: \ \xi\psi = \xi^\alpha\psi_\alpha = \xi^\alpha\psi^\beta\epsilon_{\beta\alpha} = -\xi^\alpha\epsilon_{\alpha\beta}\psi^\beta\right]$$

- Proof of Lorentz invariance

$$\xi'^\alpha = \Lambda^\alpha{}_\beta\xi^\beta, \quad \psi'^\alpha = \Lambda^\alpha{}_\beta\psi^\beta \quad (\Lambda \in SL(2,\mathbb{R}))$$

$$\rightarrow \xi'\psi' \equiv \underline{C_{\alpha\beta}\Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta}\,\xi^\gamma\xi^\delta = \underline{(\det\Lambda)\,C_{\gamma\delta}}\,\xi^\gamma\xi^\delta = \xi\psi$$

▶ The inner product is usually defined using <u>Dirac conjugate</u> $\bar{\Psi} \equiv \psi^{+} \gamma^{0}$

▶ Here, we have used the <u>Majorana conjugate</u> $\psi_\alpha \equiv \psi^\beta C_{\beta\alpha}$ [PvN: $\psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}$]

• The two are equivalent for Majorana spinors.

• <u>Simple Supersymmetry</u>: Fermionic conserved charges Q^α and $(Q^\alpha)^\dagger = Q^\alpha$ and the anticommutation relation

$$\{Q^\alpha, Q^\beta\} = -2(\gamma^a C^{-1})^{\alpha\beta} P_a$$

$$(P_0, P_1, P_2) = (-E, P^1, P^2) \quad : \text{energy-momentum}$$

• <u>Extended Supersymmetry</u>: $Q^{\alpha I} \quad (I = 1, \dots, \mathcal{N})$

$$\{Q^{\alpha I}, Q^{\beta J}\} = -2\delta^{IJ}(\gamma^a C^{-1})^{\alpha\beta} P_a$$

– <u>R-Symmetry</u>: The anticommutator is invariant under

$$Q^{\alpha I} \longrightarrow R^I{}_J \, Q^{\alpha J} \quad \text{for } R \in SO(\mathcal{N})$$

For $\mathcal{N}=2$, the R-symmetry is $SO(2) \cong U(1)$.

The $\mathcal{N}=2$ algebra can be written using the linear combinations

$$Q^\alpha \equiv \frac{1}{2}(Q^{\alpha 1} + i Q^{\alpha 2}) \qquad U(1) \text{ R-charge } -1$$

$$\bar{Q}^\alpha \equiv \frac{1}{2}(Q^{\alpha 1} - i Q^{\alpha 2}) \qquad +1$$

$$(Q^\alpha)^\dagger = \bar{Q}^\alpha \qquad (Q^{\alpha 1} \text{ \& } Q^{\alpha 2} \text{ are real})$$

– <u>3d $\mathcal{N}=2$ SUSY algebra</u>

$$\{Q^\alpha, \bar{Q}^\beta\} = -(\gamma^a C^{-1})^{\alpha\beta} P_a$$

$$\{Q^\alpha, Q^\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0$$

written in terms of Q^α and $\bar{Q}^\beta$

▸ $(\xi Q + \bar\xi \bar Q)^2 = \xi \gamma^a \bar\xi P_a$ for Grassmann-even spinors $\xi, \bar\xi$ (Grassmann even spinors commute)

Proof

$$(\xi Q)(\xi Q) = (\xi^\alpha C_{\alpha\beta} Q^\beta)(\xi^\rho C_{\rho\sigma} Q^\sigma)$$

$$= -\xi^\alpha \xi^\rho C_{\alpha\beta} C_{\rho\sigma} Q^\beta Q^\sigma$$

$$C^{\beta\mu} Q_\mu \quad C^{\sigma\nu} Q_\nu$$

$$= -\xi^\alpha \xi^\rho \underbrace{(C_{\alpha\beta} C^{\beta\mu})}_{-\delta_\alpha{}^\mu} \underbrace{(C_{\rho\sigma} C^{\sigma\nu})}_{-\delta_\rho{}^\nu} Q_\mu Q_\nu$$

$$= -\underbrace{\xi^\alpha \xi^\rho}_{\text{commute}} Q_\alpha Q_\rho$$

$$= -\frac{1}{2} \xi^\alpha \xi^\rho \{Q_\alpha, Q_\rho\} = 0$$

$$C_{\alpha\beta} C_{\rho\sigma} \stackrel{?}{=} \delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho}$$

$$C_{11} C_{11} = \delta_{11}\delta_{11} - \delta_{11}\delta_{11} \checkmark$$

$$C_{12} C_{21} = \delta_{12}\delta_{21} - \delta_{11}\delta_{22} \checkmark$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{I}$$

$$Q_\alpha = Q^\beta C_{\beta\alpha}$$

$$C^{\gamma\alpha} Q_\alpha = Q^\beta C^{\gamma\alpha} C_{\beta\alpha}$$

$$= -Q^\beta C^{\gamma\alpha} C_{\alpha\beta}$$

$$= -Q^\beta (-\delta^\gamma{}_\beta)$$

$$= Q^\gamma$$

$$Q^\alpha = C^{\alpha\beta} Q_\beta$$

similarly, $(\bar\xi \bar Q)(\bar\xi \bar Q) = 0$

The only term to consider is the cross term $2(\xi Q)(\bar\xi \bar Q)$

now, $$2(\xi Q)(\bar\xi \bar Q) = -2 \underbrace{\xi^\alpha \bar\xi^\rho}_{\text{commute}} Q_\alpha \bar Q_\rho$$

$$= -2 \xi^\alpha \bar\xi^\rho \frac{1}{2} \{Q_\alpha, \bar Q_\rho\}$$

$$= -\xi^\alpha \bar\xi^\rho (-(\gamma^a C^{-1})_{\alpha\rho}) P_a$$

$$= \xi^\alpha C^{\rho\sigma} (\gamma^a C^{-1})_{\alpha\rho} \bar\xi_\sigma P_a$$

$$= \xi^\alpha (\gamma^a C^{-1} \cdot C)_\alpha{}^\sigma \bar\xi_\sigma P_a$$

$$= \xi^\alpha (\gamma^a)_\alpha{}^\sigma \bar\xi_\sigma P_a$$

$$= \xi \gamma^a \bar\xi P_a$$

$$\bar\xi^\rho = C^{\rho\sigma} \bar\xi_\sigma$$

as was required to be shown

QED ■

3D $\mathcal{N}=2$ SUSY Theories in Euclidean Space ($\mathbb{R}^3$)

Spinor Conventions

- Gamma matrices for $SO(3)$

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1 \, , \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2 \, , \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3 \, , \quad \{\gamma^a, \gamma^b\} = 2\delta^{ab}$$

- Generators of $SO(3)$ in spinor representation : $\frac{1}{2}\gamma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$
- $\{i\gamma^{12}, i\gamma^{23}, i\gamma^{13}\}$ span the linear space of 2×2 hermite traceless matrices, so $SO(3) \cong SU(2)$

$$\left.\begin{aligned} (\gamma^1)^\dagger &= \gamma^1 \\ (\gamma^2)^\dagger &= \gamma^2 \\ (\gamma^3)^\dagger &= \gamma^3 \end{aligned}\right\} \text{ for } SO(3)$$

- Invariant inner product of spinors

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta \, , \qquad C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- We will write

$$\xi\psi \equiv \xi^\alpha C_{\alpha\beta}\psi^\beta$$
$$\xi\gamma^a\psi \equiv \xi^\alpha C_{\alpha\beta}(\gamma^a)^\beta{}_\gamma \psi^\gamma$$

- <u>Note</u> : $C_{\alpha\beta} = -C_{\beta\alpha}$ <u>but</u> $(C\gamma^a)_{\alpha\beta} = (C\gamma^a)_{\beta\alpha}$
- So, for Grassmann odd spinors, one has

$$\xi\psi = \psi\xi$$
$$\xi\gamma^a\psi = -\psi\gamma^a\xi$$

Free Wess Zumino Model in $\mathbb{R}^3$

Fields : $\phi, \bar\phi$ complex scalar

$\psi, \bar\psi$ complex spinor

Lagrangian : $L = \partial_m\bar\phi\,\partial_m\phi - i\bar\psi\gamma^m\partial_m\psi$

Transformation rules : $\delta\phi = \xi\psi$; $\delta\psi = i\gamma^m\bar\xi\,\partial_m\phi$

$\delta\bar\phi = \bar\xi\bar\psi$; $\delta\bar\psi = i\gamma^m\xi\,\partial_m\bar\phi$

Invariance of the Lagrangian under SUSY Transformation laws

$$\delta\mathcal{L} = \partial_m(\delta\bar\phi)\partial_m\phi + \partial_m\bar\phi\,\partial_m(\delta\phi) - i(\delta\bar\psi)\gamma^m\partial_m\psi - i\bar\psi\gamma^m\partial_m(\delta\psi)$$

$$= \partial_m(\bar\xi\bar\psi)\partial_m\phi + \partial_m\bar\phi\,\partial_m(\xi\psi) - i(i\gamma^n\xi\partial_n\bar\phi)\gamma^m\partial_m\psi - i\bar\psi\gamma^m\partial_m(i\gamma^n\bar\xi\partial_n\phi)$$

not partially integrand (no need)

partially integrate

$$= -\bar\xi\bar\psi\cdot\partial^2\phi - \partial^2\bar\phi\cdot\xi\psi + \partial_m\partial_n\bar\phi\cdot\xi\gamma^n\gamma^m\psi + \bar\psi\gamma^m\gamma^n\bar\xi\cdot\partial_m\partial_n\phi + \text{total derivatives}$$

now, $\gamma^m\gamma^n = \frac{1}{2}\{\gamma^m,\gamma^n\} + \frac{1}{2}[\gamma^m,\gamma^n] = \delta^{mn} + \epsilon^{mnp}\gamma_p$

$$= \frac{1}{2}\cdot 2\delta^{mn} + \frac{1}{2}\cdot 2\epsilon^{mnp}\gamma_p$$

$\therefore$ ___ $= \partial^2\bar\phi\cdot\xi\psi + \cancel{\partial_m\partial_n\bar\phi\,\epsilon^{mnp}\xi\gamma_p\psi}$

and ___ $= \bar\psi\bar\xi\cdot\partial^2\phi + \cancel{\bar\psi\gamma_p\bar\xi\,\epsilon^{mnp}\cdot\partial_m\partial_n\phi}$

the ~~~ terms and the ~~~ terms thus cancel.

$\Rightarrow \delta\mathcal{L} = 0 +$ total derivatives (surface terms on integration)

SUSY Transformation Laws

$\delta\phi = \xi\psi$, $\delta\psi = i\gamma^m\bar\xi\,\partial_m\phi$

$\delta\bar\phi = \bar\xi\bar\psi$, $\delta\bar\psi = i\gamma^m\xi\,\partial_m\bar\phi$

So the question is, can we reproduce the SUSY algebra?

$\delta^2 = (\xi Q + \bar\xi\bar Q)^2 = \xi\gamma^a\bar\xi P_a$?

($\xi, \bar\xi$ are Grassmann even here)

Check

$\delta^2\phi = \delta(\xi\psi) = \xi(i\gamma^m\bar\xi\partial_m\phi) = i\xi\gamma^m\bar\xi\partial_m\phi$ ✓

$\delta^2\psi = \delta(i\gamma^m\bar\xi\partial_m\phi) = i\gamma^m\bar\xi\partial_m(\xi\psi) = i\gamma^m\underbrace{\bar\xi\,\xi\,\partial_m\psi}$

how to proceed?

▶ use the FIERZ IDENTITY

$\boxed{\text{Fierz Identity: } \epsilon(\chi\eta) + \chi(\eta\epsilon) + \eta(\epsilon\chi) = 0}$

For us, this translates to

$$\underbrace{\underbrace{(\gamma^m\bar\xi)}_{\text{"}\epsilon\text{"}}(\underbrace{\xi}_{\text{"}\chi\text{"}}\underbrace{\partial_m\psi}_{\text{"}\eta\text{"}})}_{\text{this is the term we have in } \delta^2\psi} + \xi(\partial_m\psi\gamma^m\bar\xi) + \partial_m\psi(\underbrace{\gamma^m\bar\xi\,\xi}) = 0$$

$$\gamma^m\bar\xi\,\xi = \xi\gamma^m\bar\xi = -\bar\xi\gamma^m\xi$$

using properties listed on the prev. page so, to summarize we had $\delta^2\psi = i\gamma^m\bar{\xi}\,\xi\,\partial_m\psi$

and the Fierz identity gives us $\gamma^m\bar{\xi}\,\xi\,\partial_m\psi = -\xi(\partial_m\psi\,\gamma^m\bar{\xi}) + \partial_m\psi(\bar{\xi}\gamma^m\xi)$

so,
$$\begin{aligned}
\delta^2\psi &= -i\xi(\partial_m\psi\,\gamma^m\bar{\xi}) + i\,\partial_m\psi(\bar{\xi}\gamma^m\xi)\\
&= -i(\partial_m\psi)\,\xi\gamma^m\bar{\xi} + i(\bar{\xi}\gamma^m\xi)\,\partial_m\psi\\
&= +i(\partial_m\psi)\,\bar{\xi}\gamma^m\xi - \underbrace{\boxed{i\,\xi\gamma^m\bar{\xi}\,\partial_m\psi}}_{=0 \text{ by E.O.M.}}\\
&\overset{\text{EOM}}{=} i(\bar{\xi}\gamma^m\xi)\,\partial_m\psi
\end{aligned}$$

$\Rightarrow$ The SUSY algebra $\delta^2 = i\xi\gamma^m\bar{\xi}\partial_m$ is realized on-shell.

- The SUSY algebra can be realized off-shell by adding auxiliary fields.

$$\boxed{\begin{aligned}
&\delta\phi = \xi\psi \;;\quad \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F \;;\quad \delta F = i\bar{\xi}\gamma^m\partial_m\psi\\
&\delta\bar{\phi} = \bar{\xi}\bar{\psi} \;;\quad \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F} \;;\quad \delta\bar{F} = i\xi\gamma^m\partial_m\bar{\psi}
\end{aligned}}$$

$\delta^2 = i\xi\gamma^m\bar{\xi}\partial_m$ without assuming E.O.M.

- Free Lagrangian : $\mathcal{L} = \partial_m\phi\,\partial_m\bar{\phi} - i\bar{\psi}\gamma^m\partial_m\psi + \bar{F}F$

Invariance of the Lagrangian under SUSY transformation laws

$$\begin{aligned}
\delta\mathcal{L} &= \partial_m(\xi\psi)\,\partial_m\bar{\phi} + \partial_m\phi\,\partial_m(\bar{\xi}\bar{\psi}) - i(i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F})\gamma^n\partial_n\psi - i\bar{\psi}\gamma^m\partial_m(i\gamma^n\bar{\xi}\partial_n\phi + \xi F)\\
&\quad + (i\xi\gamma^m\partial_m\bar{\psi})F + \bar{F}(i\bar{\xi}\gamma^m\partial_m\psi)\\
&= \xi(\partial_m\psi)(\partial_m\bar{\phi}) + \bar{\xi}(\partial_m\phi)(\partial_m\bar{\psi}) + \gamma^m\xi(\partial_m\bar{\phi})\gamma^n(\partial_n\psi) - i\bar{\xi}\bar{F}\gamma^n(\partial_n\psi) + \underline{\bar{\psi}\gamma^m\gamma^n\bar{\xi}(\partial_m\partial_n\phi)}\\
&\quad + i(\bar{F}\bar{\xi}\gamma^m\partial_m\psi)\\
&\quad \underline{- i\bar{\psi}\gamma^m\xi(\partial_m F)} + i(\xi\gamma^m\partial_m\bar{\psi})F + i(\bar{\xi}\gamma^n\psi)\,\partial_n\bar{F} + \bar{\psi}\gamma^m\gamma^n\bar{\xi}\,\partial_m\partial_n\phi
\end{aligned}$$

terms underlined in black do not have to be integrated partially as they're already in the form we want

Partially integrate
$$\begin{aligned}
&= -\xi\psi\cdot\partial^2\bar{\phi} - \bar{\xi}\bar{\psi}\cdot\partial^2\phi - (\gamma^m\xi\gamma^n\psi)\,\partial^2\bar{\phi} - i(\xi\gamma^m\bar{\psi})\,\partial_m F - i(\bar{\xi}\gamma^m\psi)\,\partial_m\bar{F}\\
&\quad - i(\bar{\psi}\gamma^m\xi)\,\partial_m F + \text{total derivatives}
\end{aligned}$$

cancel

cancel exactly as they did on the previous page for the non-auxiliary field formulation

cancels because $\bar{\psi}\gamma^m\xi = -\xi\gamma^m\bar{\psi}$

$\Rightarrow \delta\mathcal{L} = 0 +$ total derivatives

SUPERMULTIPLET : set of fields on which the supersymmetry is realized irreducibly

chiral multiplet : (ϕ, ψ, F)

antichiral multiplet : $(\bar{\phi}, \bar{\psi}, \bar{F})$

Vector Multiplet

A_m : gauge field for the gauge group G

σ, D : scalars in the adjoint rep^n of G

$\lambda, \bar{\lambda}$: spinors in the adjoint rep^n of G

} Lie-algebra valued (matrices)

$F_{mn} \equiv \partial_m A_n - \partial_n A_m - i[A_m, A_n]$: field strength

$D_m \lambda \equiv \partial_m \lambda - i[A_m, \lambda]$: covariant derivative

SUSY TRANSFORMATION RULES (VECTOR MULTIPLET)

$$\delta A_m = -\frac{i}{2}(\xi \gamma_m \bar{\lambda} + \bar{\xi} \gamma_m \lambda)$$

$$\delta \sigma = \frac{1}{2}(\xi \bar{\lambda} - \bar{\xi} \lambda)$$

$$\delta D = \frac{i}{2} \xi (\gamma^m D_m \bar{\lambda} + [\sigma, \bar{\lambda}]) - \frac{i}{2} \bar{\xi} (\gamma^m D_m \lambda - [\sigma, \lambda])$$

$$\delta \lambda = \frac{1}{2} \gamma^{mn} \xi F_{mn} - \xi D - i \gamma^m \xi D_m \sigma$$

$$\delta \bar{\lambda} = \frac{1}{2} \gamma^{mn} \bar{\xi} F_{mn} + \bar{\xi} D + i \gamma^m \bar{\xi} D_m \sigma$$

SUSY TRANSFORMATION RULES (CHIRAL MULTIPLET)

$$\delta \phi = \xi \psi \quad ; \quad \delta \bar{\phi} = \bar{\xi} \bar{\psi}$$

$$\delta \psi = i \gamma^m \bar{\xi} D_m \phi + i \bar{\xi} \sigma \phi + \xi F$$

$$\delta \bar{\psi} = i \gamma^m \xi D_m \bar{\phi} + i \xi \bar{\phi} \sigma + \bar{\xi} \bar{F}$$

$$\delta F = \bar{\xi} (i \gamma^m D_m \psi - i \sigma \psi - i \bar{\lambda} \phi)$$

$$\delta \bar{F} = \xi (i \gamma^m D_m \bar{\psi} - i \bar{\psi} \sigma + i \bar{\phi} \lambda)$$

Matter Coupled to Gauge Fields

▶ The chiral multiplet fields can be coupled to gauge fields.

- (ϕ, ψ, F) belong to a rep. R of G

$$D_m \phi \equiv \partial_m \phi - i A_m \phi = \partial_m \phi - i A_m^a T_a^R \phi$$

$$D_m \psi \equiv \partial_m \psi - i A_m \psi = \partial_m \psi - i A_m^a T_a^R \psi$$

Note: The fact that the covariant derivative acts differently on fields (ϕ, ψ, F) and $(A_m, \sigma, D, \lambda)$ is known from non-supersymmetric Yang-Mills theory and should not come as a surprise.

e.g. $D_m \phi = \partial_m \phi - A_m^a T_a^R \phi$ $\qquad$ T_a^R = generator of rep^n R in which ϕ lives.

D_m "knows" how to act once it is supplied an operand.

- $(\bar{\phi}, \bar{\psi}, \bar{F})$ belong to a rep. $\bar{R}$ of G

$$D_m \bar{\phi} \equiv \partial_m \bar{\phi} + i \bar{\phi} A_m = \partial_m \bar{\phi} + i \bar{\phi} A_m^a T_a^R$$

$$D_m \bar{\psi} \equiv \partial_m \bar{\psi} + i \bar{\psi} A_m = \partial_m \bar{\psi} + i \bar{\psi} A_m^a T_a^R$$

U(1) R-charges

The R-symmetry group in 3d $\mathcal{N}=2$ is $SO(2) \simeq U(1)$.
(The supercharges in 3D transform as 2-component real spinors.)

We assign to $(\xi, \bar{\xi})$ the R-charges $(+1, -1)$.

$$\underset{\downarrow \atop R=0}{\delta A_m} = -\frac{i}{2}\left(\underset{\uparrow \atop 1}{\xi}\gamma_m \underset{\uparrow \atop -1}{\bar{\lambda}} + \underset{\uparrow \atop -1}{\bar{\xi}}\gamma_m \underset{\uparrow \atop +1}{\lambda}\right)$$

Vector multiplet

field	R-charge
A_m	0
σ	0
λ	+1
$\bar{\lambda}$	−1
D	0

Chiral multiplet

field	R-charge
ϕ	r
ψ	$r-1$
F	$r-2$

r : arbitrary

Anti-chiral multiplet

field	R-charge
$\bar{\phi}$	$-r$
$\bar{\psi}$	$-r+1$
$\bar{F}$	$-r+2$

INVARIANT LAGRANGIANS

- Yang-Mills & Chern-Simons Terms

  - $$\mathcal{L}_{YM} = \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + i\bar{\lambda}\gamma^m D_m\lambda - i\bar{\lambda}[\sigma,\lambda]\right)$$
  - $$\mathcal{L}_{CS} = \frac{ik}{4\pi}\mathrm{Tr}\left[\epsilon^{mnp}\left(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p\right) - (\bar{\lambda}\lambda + 2\sigma D)\right]$$

- Fayet-Iliopoulos term * for U(1) vector multiplet only

$$\mathcal{L}_{FI} = -\frac{i\zeta}{\pi}D$$

- Kinetic terms for chiral multiplets :

$$\mathcal{L}_{mat} = D_m\bar{\phi}D^m\phi + \bar{\phi}\sigma^2\phi - i\bar{\phi}D\phi + \bar{F}F - i\bar{\psi}\gamma^m D_m\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi$$

- F-Term for chiral multiplets (ϕ_i, ψ_i, F_i)

$$\mathcal{L}_{F\text{-Term}} = F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}$$

W : SUPERPOTENTIAL
= gauge invariant function of $\{\phi_i\}$

Remarks

- Standard trace when writing gauge invariants

$$\mathrm{Tr}(\cdots) = \frac{1}{2h^\vee} \mathrm{Tr}_{(\mathrm{adj})}(\cdots\cdot)$$

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$$= \begin{cases} \mathrm{Tr}_{(N\times N)}(\cdots\cdot) & \text{for } SU(N), USp(N) \\ \frac{1}{2}\mathrm{Tr}_{(N\times N)}(\cdots\cdot) & \text{for } SO(N) \end{cases}$$

Some background...

Source: http://member.ipmu.jp/kentaro.hori/Courses/EPP/Lgrp.pdf

▶ The dual Coxeter number

Trace in any repⁿ R of a Lie gp. G ⟶ invariant bilinear form on the Lie algebra

If G is simple, all invariant bilinear forms are proportional to one another.

We define the number T_R by the proportionality to "Tr"

e.g. $\mathrm{tr}_R(XY) = 2T_R \,\mathrm{Tr}(XY)$

$$T_R = \begin{cases} 1/2 & \text{for } R = \mathbb{C}^n \text{ of } SU(n) \\ 1 & \text{for } R = \mathbb{R}^n \text{ of } SO(n) \\ 1/2 & \text{for } R = \mathbb{C}^{2n} \text{ of } USp(2n) \end{cases}$$

T_R values for defining representations of $SU(n), SO(n)$ and $USp(2n)$

T_{adj} (T_R for R = adj) defines the dual Coxeter number $h^\vee$ of G

$$\mathrm{tr}_{adj}(XY) = 2h^\vee \,\mathrm{Tr}(XY)$$

so $$\frac{\mathrm{tr}_{adj}(XY)}{\mathrm{tr}_R(XY)} = \frac{2h^\vee}{2T_R} \Rightarrow \mathrm{tr}_R(XY) = \frac{T_R}{h^\vee}\mathrm{tr}_{adj}(XY)$$

⟨Hori⟩

Note: Hori's definition differs from Hosomichi's defⁿ

The dimension, rank and dual Coxeter number of simple Lie groups is listed below (table from Hori's notes).

G	dimension	rank	$h^\vee$
$SU(n)$	n^2-1	$n-1$	n
$SO(2m+1)$	$(2m+1)m$	m	$2m-1$
$Sp(n)$	$n(2n+1)$	n	$n+1$
$SO(2m)$	$m(2m-1)$	m	$2m-2$
E_6	78	6	12
E_7	133	7	18
E_8	248	8	30
F_4	52	4	9
G_2	14	2	4

continued

- The Chern-Simons action, $S_{CS} = \frac{ik}{4\pi}\int \mathrm{Tr}\left(A\,dA - \frac{2i}{3}A^3\right)$ is gauge invariant up to shifts by $2\pi i\mathbb{Z}$ if $k \in \mathbb{Z}$.

3D $\mathcal{N}=2$ SUSY THEORIES : A SUMMARY

- $\mathcal{N}=2$ SUSY Algebra

Two-component supercharges Q^α, $\bar{Q}^\alpha$ satisfying

$$\{Q^\alpha, \bar{Q}^\beta\} = -(\gamma^a C^{-1})^{\alpha\beta} P_a$$

- $\mathcal{N}=2$ SUSY gauge theories can be constructed from

Vector multiplet : $(A_m, \sigma, \lambda, \bar{\lambda}, D)$ gauge gp. G

Chiral multiplet : (ϕ, ψ, F) rep. R $U(1)$ R-charge r

Anti Chiral multiplet : $(\bar{\phi}, \bar{\psi}, \bar{F})$ rep. $\bar{R}$ $U(1)$ R-charge $-r$

Invariant Lagrangian : $\mathcal{L}_{YM}$, $\mathcal{L}_{CS}$, $\mathcal{L}_{FI}$, $\mathcal{L}_{mat}$, $\mathcal{L}_{F\text{-Term}}$

FIELD THEORIES IN CURVED SPACE

- The metric is curved : $ds^2 = g_{mn}(x)\, dx^m dx^n$
- No preferred choice of coordinates
  $\rightarrow$ action has to be written in a general covariant way

$$S = \int d^3x \sqrt{g}\, \mathcal{L}$$

$$\partial_m \bar{\phi}\, \partial_m \phi \longrightarrow g^{mn}\partial_m \bar{\phi}\, \partial_n \phi$$

$$\bar{\psi}\gamma^m \partial_m \psi \longrightarrow \bar{\psi}\gamma^m D_m \psi = \bar{\psi}\gamma^m \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab}\right)\psi$$

- Need to define spinors on curved spaces.
  Spinors on curved spaces transform under local Lorentz transformations.

Vielbein and local Lorentz symmetry

- Vielbein $e_m^a(x)$ is a matrix-valued field satisfying

$$g_{mn}(x) = \delta_{ab}\, e_m^a(x)\, e_n^b(x) \quad \text{— } (*)$$

$$ds^2 = g_{mn}(x)\, dx^m dx^n = \delta_{ab}\, e^a e^b \quad ; \quad e^a \equiv e_m^a\, dx^m$$
(Vielbein 1-form)

- Given $g_{mn}(x)$, the choice of $e_m^a(x)$ satisfying $(*)$ is not unique.
  Different choices are related to one another by local Lorentz transformations

$$e'^a_m(x) = \Lambda^a{}_b(x)\, e^b_m(x) \quad , \quad \Lambda^a{}_b(x) \in SO(3)$$

- Infinitesimal local Lorentz transformation :

$$\delta e_m^a(x) = w^a{}_b(x)\, e^b{}_m(x)$$

$$\delta \psi(x) = \frac{1}{4} w_{ab}(x)\gamma^{ab}\psi(x)$$

Curved and Flat Indices

- curved indices $m, n \ldots$
  transform under general coordinate transformations.
- flat indices $a, b, \ldots$
  transform under local Lorentz transformations.

Vielbein can convert one index to the other, e.g.

$\gamma^a :=$ constant matrices satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$

$\gamma^m :=$ coordinate-dependent, satisfy $\{\gamma^m, \gamma^n\} = 2g^{mn}$

$$\gamma^a = e_m^a \gamma^m$$

$$\gamma^m = e^m_a \gamma^a$$

Spin Connection & Covariant Derivative

- Covariant derivative of spinor fields

$$D_m \Psi \equiv \partial_m \Psi + \frac{1}{4} \Omega_m^{\ ab} \gamma_{ab} \Psi$$

$$\Omega_m^{\ ab} = -\Omega_m^{\ ba} \quad : \quad \text{"spin connection"}$$

is defined so that $D_m\Psi$ transforms the same way as Ψ under local Lorentz transformations.

▶ Recall that the covariant derivatives of vectors are defined as

$$D_m V^n = \partial_m V^n + \Gamma^n_{\ mk} V^k$$

$$D_m V_n = \partial_m V_n - \Gamma^k_{\ mn} V_k$$

so that $D_m V^n$, $D_m V_n$ transform covariantly under diffeomorphisms.

Levi-Civita Connection & Spin Connection

- Levi-Civita connection $\Gamma^k_{\ mn}$ is determined from $\Gamma^k_{\ mn} = \Gamma^k_{\ nm}$ and $D_k g_{mn} = 0$.

$$\Gamma^k_{\ mn} = \frac{1}{2} g^{kl} (\partial_m g_{nl} + \partial_n g_{ml} - \partial_l g_{mn})$$

▶ Recall $\quad 0 = D_k g_{mn} = \partial_k g_{mn} - \Gamma^l_{\ km} g_{ln} - \Gamma^l_{\ kn} g_{ml}$

$$= \partial_k g_{mn} - \Gamma_{n,km} - \Gamma_{m,nk}$$

$$\left.\begin{aligned} \partial_k g_{mn} &= \Gamma_{n,km} + \Gamma_{m,nk} \\ \partial_m g_{nk} &= \Gamma_{k,mn} + \Gamma_{n,mk} \end{aligned}\right) + \quad \left.\begin{aligned} \\ \partial_n g_{km} &= \Gamma_{m,nk} + \Gamma_{k,mn} \end{aligned}\right\} -$$

- Spin connection $\Omega_m^{\ ab}$ is determined from $D_m e_n^{\ a} = 0$

$$0 = D_m e_n^{\ a} = \partial_m e_n^{\ a} + \Omega^a_{\ bm} e_n^{\ b} - \Gamma^k_{\ mn} e_k^{\ a}$$

$$\Rightarrow \quad 0 = D_{[m} e_{n]}^{\ a} = \partial_{[m} e_{n]}^{\ a} + \Omega^a_{\ b[m} e_{n]}^{\ b}$$

(Antisymmetrization wrt m,n kills the Levi-Civita term)

By introducing the 1-forms $\quad e^a \equiv e_m^{\ a} dx^m$

$$\Omega^a_{\ b} \equiv \Omega^a_{\ bm} dx^m$$

one can write

$$\boxed{0 = De^a = de^a + \Omega^a_{\ b} \wedge e^b}$$

SUSY on Curved Spaces

- SUSY parameters $\xi, \bar{\xi}$ are no longer constants, but solutions to Killing Spinor equations.
- The simplest Killing spinor equation

$$D_m \xi \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma_{ab}\right)\xi = \gamma_a e^a_m \eta$$

$$D_m \bar{\xi} \equiv \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma_{ab}\right)\bar{\xi} = \gamma_a e^a_m \bar{\eta} \qquad \text{for some } \eta, \bar{\eta}$$

has solutions on the round sphere.

An important exercise: Show that $D_m \xi = \gamma_m \eta \Rightarrow \boxed{\gamma^m D_m \eta = -\frac{1}{8} R \xi}$; R : scalar curvature

Proof:

- $\gamma^{mn} D_m D_n \xi = \gamma^{mn} D_m \gamma_n \xi = \gamma^{mn}\gamma_n D_m \xi = 2\gamma^m D_m \eta$

(using $D_m e^a_n = 0$)

using $\gamma^{mn}\gamma_p = \delta^n_p \gamma^m - \delta^m_p \gamma^n$

(so $\gamma^{mn}\gamma_n = \delta^n_n \gamma^m - \delta^m_n \gamma^n = 3\gamma^m - \gamma^m = 2\gamma^m$)

- $$\gamma^{mn} D_m D_n \xi = \frac{1}{2}\gamma^{mn}[D_m, D_n]\xi$$

$$= \frac{1}{2}\gamma^{mn} \cdot \frac{1}{4}\gamma_{ab} R^{ab}_{\ \ mn}\xi \qquad \left(R^{ab}_{\ \ mn} \equiv \partial_m \Omega_n^{ab} - \partial_n \Omega_m^{ab} + \Omega^a_{\ cm}\Omega_n^{cb} - \Omega^a_{\ cn}\Omega_m^{cb}\right)$$

Now, $\gamma^{ab}\gamma^{cd} = \gamma^{abcd} + \left(\delta^{bc}\gamma^{ad} - \delta^{ac}\gamma^{bd} - \delta^{bd}\gamma^{ac} + \delta^{ad}\gamma^{bc}\right) - \left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right)$

Recall some pertinent properties of the Riemann curvature tensor:

1. Skew symmetry: $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
2. Interchange symmetry: $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$
3. First Bianchi identity: $R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0$
4. Second Bianchi identity: $R_{\mu[\nu\rho\sigma]} = 0$

$$\therefore \gamma^{mn}\gamma^{ab}R^{ab}_{mn} = \left\{ \gamma^{mnab} + \left(\delta^{na}\gamma^{mb} - \delta^{nb}\gamma^{ma} - \delta^{ma}\gamma^{nb} + \delta^{mb}\gamma^{na}\right) - \left(\delta^{ma}\delta^{nb} - \delta^{mb}\delta^{na}\right) \right\} R^{ab}_{mn}$$

$$= \underbrace{\gamma^{mnab}R^{ab}_{mn}} + \underbrace{\left(\gamma^{mb}R^{ab}_{ma} - \gamma^{ma}R^{ab}_{mb} - \gamma^{nb}R^{ab}_{an} + \gamma^{na}R^{ab}_{bn}\right)} - \underbrace{\left(R^{ab}_{ab} - R^{ab}_{ba}\right)}$$

- Now, note that m, n are curved indices while a, b are flat. So to write this properly, we should really use vielbeins. However, we do know the explicit form of R^{mn}_{ab}. Before we use it, lets cheat a bit and use the symmetry properties of the Riemann tensor.

- Also, we should be careful about index placement. Specifically, Ω^{ab}_{m} is really $\Omega_m{}^{ab}$ and R^{ab}_{mn} is really $R_{mn}{}^{ab}$.

$$R^{ab}_{ma} := R_{ma}{}^{ab} = R_{maab} \quad \text{(we're in Euclidean space, so flat indices are lowered with } \delta^{ab}\text{)}$$

$$R^{ab}_{mb} := R_{mb}{}^{ab} = R_{mbab}$$

$$R^{ab}_{an} := R_{an}{}^{ab} = R_{anab}$$

$$R^{ab}_{bn} := R_{bn}{}^{ab} = R_{bnab}$$

$$\therefore \gamma^{mb}R^{ab}_{ma} = \gamma^{mb}R_{maab} = \gamma^{mb}R_{abma}$$

$$\gamma^{ma}R^{ab}_{mb} = \gamma^{ma}R_{mbab} = \gamma^{mb}R_{maba} = \gamma^{mb}R_{bama} = -\gamma^{mb}R_{abma}$$

$$\gamma^{nb}R^{ab}_{an} = \gamma^{nb}R_{anab} = \gamma^{mb}R_{amab} = \gamma^{mb}R_{abam} = -\gamma^{mb}R_{abma}$$

$$\gamma^{na}R^{ab}_{bn} = \gamma^{na}R_{bnab} = \gamma^{mb}R_{amba} = \gamma^{mb}R_{baam} = -\gamma^{mb}R_{abam} = \gamma^{mb}R_{abma}$$

So, $\underbrace{\qquad\qquad}$ $= 4\gamma^{mb}R_{abma} = 4\gamma^{mb}R_{baam} = 2\gamma^{mb}\left(R_{baam} - R_{maab}\right)$

$$= 2\gamma^{mb}\left(R_{baam} - R_{abma}\right)$$

$$= 2\gamma^{mb}\left(R_{baam} + R_{bama}\right)$$

Bianchi identity

$$= 2\gamma^{mb}\left(-R_{bmaa}\right)$$

$$= 0$$

(because of skew symmetry in the last 2 indices)

$$\bullet\ \underbrace{\gamma^{mnab} R_{mn}{}^{ab}} = \gamma^{mnab} R_{mnab}$$

Bianchi identity

$$= \gamma^{mnab}\left(-R_{mabn} - R_{mbna}\right)$$

$$= \gamma^{mnab} R_{ambn} + \gamma^{mnab} R_{mban}$$

$$= \gamma^{amnb} R_{ambn} + \gamma^{mbna} R_{mban}$$

$$\gamma^{ambn} = -\gamma^{mabn} = +\gamma^{manb}$$

$$= -\gamma^{ambn} R_{ambn} - \gamma^{mban} R_{mban}$$

$$= -2\gamma^{ambn} R_{ambn} = -2\gamma^{manb} R_{ambn}$$

$$= -2\gamma^{manb} R_{manb}$$

$$= -2\gamma^{mpcb}\, e_p^{\,a} e_n^{\,c}\, R_{manb}$$

$$= -2\gamma^{mpcb} R_{mpcb} = -2\gamma^{mnab} R_{mnab} \Rightarrow \gamma^{mnab} R_{mn}{}^{ab} = 0$$

I guess this is too long-winded. Maybe there's a slicker way.

So, finally,

$$\bullet\ \underbrace{\qquad\qquad} = R_{ab}{}^{ab} - R_{ba}{}^{ab}$$

$$= R_{ab}{}^{ab} - R_{ba}{}^{ab} = 2R_{ab}{}^{ab} = 2\, e^m_{\ a} e^n_{\ b} R_{mn}{}^{ab} = 2R \qquad \left(R \equiv e^m_{\ a} e^n_{\ b} R_{mn}{}^{ab}\right)$$

So,

$$\left.\begin{aligned} &\gamma^{mn} D_m D_n \xi = 2\gamma^m D_m \eta \\ \text{and}\quad &\gamma^{mn} D_m D_n \xi = \frac{1}{8}(-2R) = -\frac{1}{4}R \end{aligned}\right\} \Rightarrow \boxed{\gamma^m D_m \eta = -\frac{1}{8} R \xi}$$

Free Wess Zumino Model Revisited

$$D_m\gamma^n = D_m(\gamma^a e^n_a) = \gamma^a D_m e^n_a = 0$$

Q. Is the simple general covariantization

$$\mathcal{L} \equiv g^{mn}\partial_m\bar\phi\,\partial_n\phi - i\bar\psi\gamma^m D_m\psi + \bar F F$$

invariant under the following?

$$\delta\phi = \xi\psi\,,\quad \delta\psi = i\gamma^m\bar\xi\,\partial_m\phi + \xi F\,,\quad \delta F = i\bar\xi\gamma^m D_m\psi$$

$$\delta\bar\phi = \bar\xi\bar\psi\,,\quad \delta\bar\psi = i\gamma^m\xi\,\partial_m\bar\phi + \bar\xi\bar F\,,\quad \delta\bar F = i\xi\gamma^m D_m\bar\psi$$

— : cancel = : partial integration

$$\delta\mathcal{L} = g^{mn}\partial_m(\bar\xi\bar\psi)\partial_n\phi + g^{mn}\partial_m\bar\phi\,\partial_n(\xi\psi) - i(-i\partial_n\bar\phi\,\xi\gamma^n + \bar F\xi)\gamma^m D_m\psi - i\bar\psi\gamma^m D_m(i\gamma^n\bar\xi\,\partial_n\phi + \xi F)$$
$$+ (i\xi\gamma^m D_m\bar\psi)F + \bar F(i\bar\xi\gamma^m D_m\psi)$$

$$= -g^{mn}\bar\xi\bar\psi D_mD_n\phi - g^{mn}(D_nD_m\bar\phi)(\xi\psi) - \partial_n\bar\phi\,\xi\gamma^n\gamma^m D_m\psi + \bar\psi\gamma^m D_m(\gamma^n\bar\xi\,\partial_n\phi)$$

$$= -(\bar\xi\bar\psi)g^{mn}(D_mD_n\phi) + (D_mD_n\bar\phi)\xi\gamma^n\gamma^m\psi + \partial_n\bar\phi(D_m\xi)\gamma^n\gamma^m\psi$$
$$-(\xi\psi)g^{mn}(D_nD_m\bar\phi) + \bar\psi\gamma^m\gamma^n(D_m\bar\xi)\partial_n\phi + \bar\psi\gamma^m\gamma^n\bar\xi(D_mD_n\phi)$$

$D_mD_n\phi$ is symmetric in m & n so it picks out the symmetric part of $\gamma^m\gamma^n$, which is $\gamma^{(m}\gamma^{n)} = \frac{1}{2}\{\gamma^m,\gamma^n\} = \frac{1}{2}\cdot 2g^{mn} = g^{mn}$

also, $D_m\xi = \gamma_m\eta$, $D_m\bar\xi = \gamma_m\bar\eta$

$$= \cancel{-(\bar\xi\bar\psi)g^{mn}(D_mD_n\phi)} + \cancel{(D_mD_n\bar\phi)\xi g^{mn}\psi} + (\partial_n\bar\phi)(\gamma_m\eta)\gamma^n\gamma^m\psi$$
$$\cancel{-(\xi\psi)g^{mn}(D_nD_m\bar\phi)} + \bar\psi\gamma^m\gamma^n(\gamma_m\bar\eta)\partial_n\phi + \cancel{\bar\psi g^{mn}\bar\xi(D_mD_n\phi)}$$

$$= (\partial_n\bar\phi)\gamma_m\eta\,\gamma^n\gamma^m\psi + \bar\psi\underbrace{(\gamma^m\gamma^n\gamma_m)}_{=-\gamma^n}\bar\eta\,\partial_n\phi$$

FIX THIS

finally,

$$\gamma_m\gamma^n\gamma^m = \gamma_m(2g^{mn} - \gamma^m\gamma^n) = 2\gamma^n - 3\gamma^n = -\gamma^n$$

$$\gamma^m\gamma^n\gamma_m = (2g^{mn} - \gamma^n\gamma^m)\gamma_m = 2\gamma^n - 3\gamma^n = -\gamma^n$$

$$\therefore \delta\mathcal{L} = -(\partial_n\bar\phi)\psi\gamma^n\eta - (\partial_n\phi)\bar\psi\gamma^n\bar\eta$$

> not a total derivative (so we need to add something more...)

$$\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\,\partial_n\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$$

$$\delta\phi = \xi\psi, \qquad \delta\psi = i\gamma^m\bar{\xi}\,\partial_m\phi + \xi F + i\bar{\eta}\phi, \qquad \delta F = i\bar{\xi}\gamma^m D_m\psi$$

$$\delta\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \delta\bar{\psi} = i\gamma^m\xi\,\partial_m\bar{\phi} + \bar{\xi}\bar{F} + i\eta\bar{\phi}, \qquad \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi}$$

$$\delta\mathcal{L} = g^{mn}\partial_m(\delta\bar{\phi})\,\partial_n\phi - i(\delta\bar{\psi})\gamma^m D_m\psi + (\delta\bar{F})F + g^{mn}\partial_m\bar{\phi}\,\partial_n(\delta\phi) - i\bar{\psi}\gamma^m D_m(\delta\psi) + \bar{F}(\delta F)$$

this calculation differs from the one on the previous page only by the inclusion of the ~ terms in the SUSY transformation laws of ψ & $\bar{\psi}$. so $\delta\mathcal{L}$ is the same as that on the previous page PLUS the effect of these ~ terms.

$$= -\partial_n\bar{\phi}\,\psi\gamma^n\eta - \partial_n\phi\,\bar{\psi}\gamma^n\bar{\eta} - i(i\eta\bar{\phi})\gamma^m D_m\psi - i\bar{\psi}\gamma^m D_m(i\bar{\eta}\phi)$$

$$= -(D_n\bar{\phi})\psi\gamma^n\eta - (D_n\phi)\bar{\psi}\gamma^n\bar{\eta} + \eta\bar{\phi}\gamma^m D_m\psi + \bar{\psi}\gamma^m D_m(\bar{\eta}\phi)$$

$$= \underbrace{\bar{\phi}\,D_n(\psi\gamma^n\eta)}_{\text{Partially integrate}} + \eta\bar{\phi}\gamma^m D_m\psi + \bar{\psi}\gamma^m(D_m\bar{\eta})\phi + \bar{\psi}\gamma^m\bar{\eta}(D_m\phi) - \bar{\psi}\gamma^n\bar{\eta}(D_n\phi)$$

cancel

$$= \bar{\phi}(D_n\psi)\gamma^n\eta + \bar{\phi}\psi(D_n\gamma^n)\eta + \bar{\phi}\psi\gamma^n(D_n\eta) + \eta\bar{\phi}\gamma^m D_m\psi + \bar{\psi}\gamma^m(D_m\bar{\eta})\phi$$

now, $(D_n\psi)\gamma^n\eta = -\eta\gamma^n D_n\psi$ so the ~ terms cancel each other

$$= \bar{\phi}\psi\gamma^m D_m\eta + \phi\bar{\psi}\gamma^m D_m\bar{\eta}$$

using $\gamma^m D_m\eta = -\frac{1}{8}R\xi$ (c.f. Pg. 15), $\gamma^m D_m\bar{\eta} = -\frac{1}{8}R\bar{\xi}$

$$= -\frac{1}{8}R(\bar{\phi}\psi\xi + \phi\bar{\psi}\bar{\xi})$$

using $\delta\phi = \xi\psi$ and $\delta\bar{\phi} = \bar{\xi}\bar{\psi}$

$$= -\frac{1}{8}R\,\delta(\bar{\phi}\phi)$$

$$\Rightarrow \boxed{\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\,\partial_n\phi + \underbrace{\frac{1}{8}R\bar{\phi}\phi}_{\text{added term}} - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F}$$ is invariant.

Wess Zumino Model on Curved Space : Summary

- Lagrangian $\mathcal{L} \equiv g^{mn}\partial_m\bar{\phi}\,\partial_n\phi + \frac{1}{8}R\bar{\phi}\phi - i\bar{\psi}\gamma^m D_m\psi + \bar{F}F$

- SUSY Transformations

$$\boxed{\begin{array}{lll} \delta\phi = \xi\psi, & \delta\psi = i\gamma^m\bar{\xi}\partial_m\phi + \xi F + \underline{\frac{i}{3}\gamma^m D_m\bar{\xi}\,\phi}, & \delta F = i\bar{\xi}\gamma^m D_m\psi \\ \delta\bar{\phi} = \bar{\xi}\bar{\psi}, & \delta\bar{\psi} = i\gamma^m\xi\partial_m\bar{\phi} + \bar{\xi}\bar{F} + \underline{\frac{i}{3}\gamma^m D_m\xi\,\bar{\phi}}, & \delta\bar{F} = i\xi\gamma^m D_m\bar{\psi} \end{array}}$$

Note:

$D_m\xi = \gamma_m\eta$

$\Downarrow$

$\gamma^m D_m\xi = \gamma^m\gamma_m\eta$
$= 3\eta$

$\Rightarrow \frac{1}{3}\gamma^m D_m\xi = \eta$

$\rightarrow$ explains the $\frac{1}{3}\gamma^m D_m\xi$ term in $\delta\psi$

where $\xi, \bar{\xi}$ satisfy the Killing spinor equation

$$D_m\xi = \gamma_m\eta$$
$$D_m\bar{\xi} = \gamma_m\bar{\eta} \qquad \text{for some } \eta, \bar{\eta}$$

Generalization of Killing Spinors

- Let $g_{mn}(x)$ be a metric on a 3D space M

  $V_m(x)$, $U_m(x)$: vector fields on M

  $M(x)$: scalar field on M

- The background $\{g_{mn}, V_m, U_m, M\}$ is supersymmetric if

$$\boxed{\begin{aligned} D_m \xi &\equiv \left(\partial_m + \frac{1}{4}\Omega_m{}^{ab}\gamma_{ab} - iV_m\right)\xi = iM\gamma_m \xi - iU_m \xi - \frac{1}{2}\epsilon_{mnp} U^n \gamma^p \xi \\ D_m \tilde{\xi} &\equiv \left(\partial_m + \frac{1}{4}\Omega_m{}^{ab}\gamma_{ab} + iV_m\right)\tilde{\xi} = iM\gamma_m \tilde{\xi} + iU_m \tilde{\xi} + \frac{1}{2}\epsilon_{mnp} U^n \gamma^p \tilde{\xi} \end{aligned}}$$

have solutions.

- We restrict to the case $U_m = 0$ in the following.
- V_m is the gauge field for $U(1)$ R-symmetry.

Where did these equations come from?

The origin of the Killing spinor equation

$\{g_{mn}, V_m, U_m, M\}$ are the fields in the 3D $\mathcal{N}=2$ supergravity multiplet.

Supergravity: ... a theory of the graviton g_{mn}, gravitino ψ_m, $\bar{\psi}_m$ and other fields which is invariant under local SUSY (ξ, $\bar{\xi}$ are arbitrary functions)

$$\delta e_m^a = \xi \gamma^a \bar{\psi}_m + \bar{\xi}\gamma^a \psi_m$$ <possible typo in Hosomichi's lectures.>

$$\delta \psi_m = \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - iV_m\right)\xi - iM\gamma_m \xi + iU_m \xi + \frac{1}{2}\epsilon_{mnp} U^n \gamma^p \xi$$

$$\delta \bar{\psi}_m = \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + iV_m\right)\xi - iM\gamma_m \bar{\xi} - iU_m \bar{\xi} - \frac{1}{2}\epsilon_{mnp} U^n \gamma^p \bar{\xi}$$

So, the Killing spinor equation is: $\delta\psi_m = 0$, $\delta\bar{\psi}_m = 0$

Rigid SUSY on Curved Backgrounds

Proposal by Festuccia and Seiberg

- an off-shell local SUSY theory of gravity multiplet $\{g_{mn}, V_m, U_m, M ; \psi_m, \overline{\psi}_m\}$ coupled to vector and chiral multiplets is known.
- There is a classical configuration $\{g_{mn}, V_m, U_m, M\}$ (and $\psi_m = \overline{\psi}_m = 0$) such that $\delta\psi_m = \delta\overline{\psi}_m = 0$ has solutions $(\xi, \overline{\xi})$.

▶ Then the Lagrangian of the gravity and other multiplets

$$\mathcal{L}\left(\{g_{mn}, \ldots\}, \{A_m, \ldots\}, \{\phi, \ldots\}\right)$$

- is invariant under the above δ,
- and the above δ does not change the value of gravity multiplet fields.

<u>Justification</u> $\quad \delta\mathcal{L} = \dfrac{\delta\mathcal{L}}{\delta g_{mn}}\delta g_{mn} + \dfrac{\delta\mathcal{L}}{\delta V_m}\delta V_m + \dfrac{\delta\mathcal{L}}{\delta U_m}\delta U_m + \dfrac{\delta\mathcal{L}}{\delta M}\delta M + \dfrac{\delta\mathcal{L}}{\delta\psi_m}\delta\psi_m + \dfrac{\delta\mathcal{L}}{\delta\overline{\psi}_m}\delta\overline{\psi}_m$

The solution to the Killing spinor equations is obtained by setting the fermions to zero (which makes $\delta g_{mn} = \delta V_m = \delta U_m = \delta M = 0$) and solving $\delta\psi_m = 0$ and $\delta\overline{\psi}_m = 0$. Clearly, $\delta\mathcal{L}$ evaluated on the solution to the Killing spinor equations yields zero. So the Lagrangian is indeed invariant under the above δ.

Also, $\delta(\text{gravity multiplet fields}) = 0$.

3D $\mathcal{N}=2$ SUSY on General Curved Spaces

Vector Multiplet

$$\delta A_m = -\frac{i}{2}\left(\xi\gamma_m\bar\lambda + \bar\xi\gamma_m\lambda\right)$$

$$\delta\sigma = \frac{1}{2}\left(\xi\bar\lambda - \bar\xi\lambda\right)$$

$$\delta D = \frac{i}{2}\xi\left(\gamma^m D_m\bar\lambda + [\sigma,\bar\lambda] + \underline{iM\bar\lambda}\right) - \frac{i}{2}\bar\xi\left(\gamma^m D_m\lambda - [\sigma,\lambda] + \underline{iM\lambda}\right)$$

$$\delta\lambda = \frac{1}{2}\gamma^{mn}\xi F_{mn} - \xi D - i\gamma^m\xi D_m\sigma$$

$$\delta\bar\lambda = \frac{1}{2}\gamma^{mn}\bar\xi F_{mn} + \bar\xi D + i\gamma^m\bar\xi D_m\sigma$$

where

$$D_m\lambda \equiv \partial_m\lambda + \frac{1}{4}\Omega_m{}^{ab}\gamma^{ab}\lambda - iV_m\lambda$$

Chiral Multiplet of R-charge r

$$\delta\phi = \xi\psi\ , \quad \delta\psi = i\gamma^m\bar\xi D_m\phi + i\bar\xi\sigma\phi - \underline{2rM\bar\xi\phi} + \xi F$$

$$\delta\bar\phi = \bar\xi\bar\psi\ , \quad \delta\bar\psi = i\gamma^m\xi D_m\bar\phi + i\xi\bar\phi\sigma - \underline{2rM\xi\bar\phi} + \bar\xi\bar F$$

$$\delta F = \bar\xi\left\{i\gamma^m D_m\psi - i\sigma\psi - i\bar\lambda\phi + \underline{(2r-1)M\psi}\right\}$$

$$\delta\bar F = \xi\left\{i\gamma^m D_m\bar\psi - i\bar\psi\sigma + i\bar\phi\lambda + \underline{(2r-1)M\bar\psi}\right\}$$

where

$$D_m\phi \equiv \partial_m\phi - iA_m\phi - irV_m\phi$$

$$D_m\bar\phi \equiv \partial_m\bar\phi + i\bar\phi A_m + irV_m\bar\phi$$

Invariant Lagrangians

$$\mathcal{L}_{YM} = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}F_{mn}F^{mn} + D_m\sigma D^m\sigma + D^2 + i\bar\lambda\gamma^m D_m\lambda - i\bar\lambda[\sigma,\lambda] - M\bar\lambda\lambda\right]$$

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\mathrm{Tr}\left[\epsilon^{mnp}\left(A_m\partial_n A_p - \frac{2i}{3}A_mA_nA_p\right) - \bar\lambda\lambda - 2\sigma D - 4M\sigma^2\right]$$

$$\mathcal{L}_{FI} = -\frac{i\zeta}{\pi}(D+4M\sigma)$$

$$\mathcal{L}_{mat} = D_m\bar\phi D^m\phi + \bar\phi\sigma^2\phi + 4i(r-1)M\bar\phi\sigma\phi - 2r(2r-1)M^2\bar\phi\phi + \frac{rR}{4}\bar\phi\phi - i\bar\phi D\phi + \bar F F - i\bar\psi\gamma^m D_m\psi + i\bar\psi\sigma\psi - (2r-1)M\bar\psi\psi + i\bar\psi\bar\lambda\phi - i\bar\phi\lambda\psi$$

$$\mathcal{L}_{F\text{-Term}} = F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + \text{h.c.}$$

W : gauge-invariant function of ϕ_i of R-charge 2.

SUSY on Curved Spaces : Summary

- SUSY parameters $\xi, \bar{\xi}$ on curved spaces are not constant but solutions to Killing spinor equations.
- The general form of the Killing spinor equation has origin in supergravity. It can depend on the curved metric as well as other fields in the gravity multiplet.
- The Lagrangians and transformation rules can be generalized from flat to curved spaces.

Geometry of the 3-Sphere

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

in $\mathbb{R}^4$ with metric

$$ds^2 = l^2 (dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$$

$$x_0 = \cos\theta \cos\varphi$$
$$x_3 = \cos\theta \sin\varphi$$
$$x_2 = \sin\theta \cos\chi$$
$$x_1 = \sin\theta \sin\chi$$

$$dx_0 = -\sin\theta\cos\varphi\, d\theta - \cos\theta \sin\varphi\, d\varphi$$
$$dx_3 = -\sin\theta \sin\varphi\, d\theta + \cos\theta\cos\varphi\, d\varphi$$
$$dx_2 = \cos\theta\cos\chi\, d\theta - \sin\theta \sin\chi\, d\chi$$
$$dx_3 = \cos\theta \sin\chi\, d\theta + \sin\theta\cos\chi\, d\chi$$

- symmetry : $SO(4) \simeq SU(2)_L \times SU(2)_R$
- metric : $ds^2 = l^2 (\cos^2\theta\, d\varphi^2 + \sin^2\theta\, d\chi^2 + d\theta^2)$
  $= e^1 e^1 + e^2 e^2 + e^3 e^3$
- vielbein 1-forms : $e^1 = l\cos\theta\, d\varphi$, $e^2 = l\sin\theta\, d\chi$, $e^3 = l\, d\theta$
- spin connection : solve $de^a + \Omega^{ab}\wedge e^b = 0$

$$-l\sin\theta\, d\theta\wedge d\varphi + \Omega^{12}\wedge l\sin\theta\, d\chi + \Omega^{13}\wedge l\, d\theta = 0 \quad - ①$$

- $de^1 + \Omega^{12}\wedge e^2 + \Omega^{13}\wedge e^3 = 0 \Rightarrow$
$$l\cos\theta\, d\theta\wedge d\chi - \Omega^{12}\wedge l\cos\theta\, d\varphi + \Omega^{23}\wedge l\, d\theta = 0 \quad - ②$$
- $de^2 + \Omega^{21}\wedge e^1 + \Omega^{23}\wedge e^3 = 0 \Rightarrow$
$$-\Omega^{13}\wedge l\cos\theta\, d\varphi - \Omega^{23}\wedge l\sin\theta\, d\chi = 0 \quad - ③$$
- $de^3 + \Omega^{31}\wedge e^1 + \Omega^{32}\wedge e^2 = 0 \Rightarrow$

$\boxed{\Omega^{12} = 0}$, so from ①, $(\sin\theta\, d\varphi + \Omega^{13})\wedge d\theta = 0 \Rightarrow \boxed{\Omega^{13} = -\sin\theta\, d\varphi}$

From ②, $(\Omega^{23} - \cos\theta\, d\chi)\wedge d\theta = 0 \Rightarrow \boxed{\Omega^{23} = \cos\theta\, d\chi}$

I think there's a typo in Hosomichi's notes on pg 43

3-sphere and SU(2)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- 3-sphere is the group manifold SU(2)

$$g \equiv \begin{pmatrix} x_0 + i x_3 & i x_1 + x_2 \\ i x_1 - x_2 & x_0 - i x_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\, e^{i\varphi} & \sin\theta\, e^{i\chi} \\ -\sin\theta\, e^{-i\chi} & \cos\theta\, e^{-i\varphi} \end{pmatrix} \in SU(2)$$

$(\det(g) = 1)$

- metric : $ds^2 = l^2 dx_a dx_a$

$$= \frac{l^2}{2} \operatorname{tr}(dg^\dagger dg) = -\frac{l^2}{2} \operatorname{tr}(g^{-1}dg)^2$$

Proof : $dg = \begin{pmatrix} dx_0 + i dx_3 & i dx_1 + dx_2 \\ i dx_1 - dx_2 & dx_0 - i dx_3 \end{pmatrix}$

$$dg^\dagger dg = \begin{pmatrix} dx_0 - i dx_3 & -i dx_1 - dx_2 \\ -i dx_1 + dx_2 & dx_0 + i dx_3 \end{pmatrix} \begin{pmatrix} dx_0 + i dx_3 & i dx_1 + dx_2 \\ i dx_1 - dx_2 & dx_0 - i dx_3 \end{pmatrix}$$

$$= \begin{pmatrix} dx_0^2 + dx_3^2 + dx_2^2 + dx_1^2 & 0 \\ 0 & dx_0^2 + dx_3^2 + dx_2^2 + dx_1^2 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} \cos\theta\, e^{-i\varphi} & -\sin\theta\, e^{i\chi} \\ +\sin\theta\, e^{-i\chi} & \cos\theta\, e^{i\varphi} \end{pmatrix} = \begin{pmatrix} x_0 - i x_3 & -i x_1 - x_2 \\ -i x_1 + x_2 & x_0 + i x_3 \end{pmatrix}$$

$\blacksquare$

I'm basically too lazy to compute $(g^{-1}dg)^2$ now.

▶ Left-Invariant 1-forms : $g^{-1}dg = i\gamma^a \mu^a$

$$\mu^1 = -\sin(\varphi - \chi)\, d\theta + \frac{1}{2}\cos(\varphi - \chi) \sin 2\theta\, (d\varphi + d\chi)$$
$$\mu^2 = \cos(\varphi - \chi)\, d\theta + \frac{1}{2}\sin(\varphi - \chi) \sin 2\theta\, (d\varphi + d\chi)$$
$$\mu^3 = \frac{1}{2}(d\varphi - d\chi) + \frac{1}{2}\cos 2\theta\, (d\varphi + d\chi)$$

- Convenient choice of vielbein : $ds^2 = l^2 \mu^a \mu^a$; $e^a \equiv l \mu^a$.

Let us determine the spin connection from

$$de^a + \Omega^{ab} \wedge e^b = 0 \qquad (e^a = l\mu^a)$$

Now, $i\gamma^a \mu^a = g^{-1}dg$

$\therefore\ i\gamma^a d\mu^a = d(g^{-1}dg)$

$$g^{-1}g = 1 \Rightarrow d(g^{-1}g) = 0$$
$$\Rightarrow dg^{-1} \cdot g + g^{-1}dg = 0$$
$$\Rightarrow dg^{-1} = -g^{-1}dg\, g^{-1}$$

so,

$$i\gamma^a d\mu^a = (-g^{-1} \cdot dg \cdot g^{-1})\, dg$$
$$= -g^{-1}dg \cdot g^{-1}dg$$
$$= -(i\gamma^a \mu^a)^2 = (\gamma^a \mu^a)^2$$
$$= \gamma^{ab} \mu^a \wedge \mu^b$$
$$= i\epsilon^{abc} \gamma^c \mu^a \wedge \mu^b \quad c \leftrightarrow a$$
$$= i\epsilon^{cba} \gamma^a \mu^c \wedge \mu^b$$
$$= i\epsilon^{acb} \gamma^a \mu^c \wedge \mu^b$$
$$= i\epsilon^{abc} \gamma^a \mu^b \wedge \mu^c$$

jump directly in the talk

$$\therefore\ d\mu^a = \epsilon^{abc} \mu^b \wedge \mu^c$$

$$de^a = l\, d\mu^a = l\, \epsilon^{abc} \mu^b \wedge \mu^c$$

$$\therefore\ l\, \epsilon^{abc} \mu^b \wedge \mu^c + \Omega^{ab} \wedge l \mu^b = 0$$

$$\Rightarrow (\Omega^{ab} - \epsilon^{abc} \mu^c) \wedge \mu^b = 0$$

$$\Rightarrow \boxed{\Omega^{ab} = \epsilon^{abc} \mu^c = \frac{1}{l} \epsilon^{abc} e^c}$$

Killing Spinors

Constant spinors satisfy $d\xi = 0$

$$D\xi = d\xi + \frac{1}{4}\gamma^{ab}\Omega^{ab}\xi$$ (suppressing the index on D) see below

$$= 0 + \frac{1}{4}\gamma^{ab}\left(\frac{1}{l}\epsilon^{abc}e^c\right)\xi$$

$$= \frac{1}{4l}\epsilon^{abc}\gamma^{ab}e^c\xi$$

$$[\gamma^a,\gamma^b] = 2\gamma^{ab} = 2i\epsilon^{abc}\gamma^c$$

$$= \frac{1}{4l}(2i\gamma^c)e^c\xi$$

$$\therefore \gamma^{ab} = i\epsilon^{abc}\gamma^c$$

$$= \frac{i}{2l}\gamma^c e^c\xi$$

$$\Rightarrow \epsilon^{abd}\gamma^{ab} = i\,2\delta^{cd}\gamma^c = 2i\gamma^d$$

$$\therefore \boxed{D\xi = \frac{i}{2l}\gamma^c e^c\xi} \quad \text{or} \quad \boxed{D_m\xi = \frac{i}{2l}\gamma^c e^c{}_m\xi}$$

$$\boxed{dx^m D_m\xi = \frac{i}{2l}\gamma^a(dx^m e^a{}_m)\xi} \qquad \Downarrow \qquad \boxed{D_m\xi = \frac{i}{2l}\gamma_m\xi}$$

The round sphere S^3 of radius l with the background fields $M = \frac{1}{2l}$, $U_m = V_m = 0$ has 2 Killing spinors for both ξ and $\bar{\xi}$.

Killing Vectors

- Define $R^a \equiv R^{am}\frac{\partial}{\partial x^m}$ by the property $R^a g = i g\gamma^a$

$$g \equiv \begin{pmatrix} \cos\theta\, e^{i\varphi} & \sin\theta\, e^{i\chi} \\ -\sin\theta\, e^{-i\chi} & \cos\theta\, e^{-i\varphi} \end{pmatrix}$$

$$R^1 = -\sin(\varphi-\chi)\partial_\theta + \cos(\varphi-\chi)\left[\tan\theta\,\partial_\theta + \cot\theta\,\partial_\chi\right]$$

$$R^2 = +\cos(\varphi-\chi)\partial_\theta + \sin(\varphi-\chi)\left[\tan\theta\,\partial_\varphi + \cot\theta\,\partial_\chi\right]$$

$$R^3 = \partial_\varphi - \partial_\chi$$

Important properties

(1) $\frac{1}{2i}R^a$ satisfies $SU(2)$ commutation relations.

(2) $R^{am}\mu^b{}_m = \delta^{ab}$ ($R^{am} \sim$ inverse vielbein)

Proof:

$$R^{am}\mu^b{}_m = R^{am}\left(-\frac{i}{2}\mathrm{Tr}\left[g^{-1}\partial_m g\gamma^b\right]\right)$$

$$= -\frac{i}{2}\mathrm{Tr}\left[g^{-1}R^a g\gamma^b\right]$$

$$= -\frac{i}{2}\mathrm{Tr}\left[g^{-1}ig\gamma^a\gamma^b\right]$$

$$= \frac{1}{2}\mathrm{Tr}\left[\gamma^a\gamma^b\right]$$

$$= \frac{1}{2}\cdot 2\delta^{ab} = \delta^{ab}$$

∎

Geometry of 3-sphere: Summary

- Round sphere of radius l with the background fields

  $M = \frac{1}{2l}$, $U_m = V_m = 0$ is supersymmetric.

- Round sphere is $SO(4) \sim SU(2) \times SU(2)$ - symmetric

Metric, vielbein, Laplace operator, Dirac operator .... can be expressed in terms of μ^a, Q^a, which transform nicely under the $SU(2)$.

- The path integral for sphere partition function can be explicitly performed using SUSY localization.

  We only need the representation theory of $SU(2)$.