Notes on Boundary Conformal Field Theory

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Abstract

These are notes on Boundary Conformal Field Theory, prepared for two talks in the YITP Graduate Seminar, on February 19 and February 26, 2016.

Contents

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1 Outline

Boundary Conformal Field Theory (BCFT) is the study of conformal field theory on domains with a boundary. It turns out that a boundary introduces certain constraints ("conformal boundary conditions") which have to be accounted for in formulating the theory. The overview of this talk is as follows.

- **Surface critical behavior of the Ising Model** (a toy example): we will look at the mean field theory approach to the Ising Model, starting from the self-consistent field equation obtained by minimizing the free-energy. We will see how mean field theory predicts the ordinary and extraordinary transitions in the system, and how one can extract bulk and surface critical exponents. The sytem exhibits conformal invariance at criticality, which motivates our study of the simplest CFT system: the two-dimensional case.
- **Two dimensional CFT**: This will be brief review, mostly to establish context and notation. A special object of interest will be the correlation function of the stress tensor with primary fields, which obeys the so called Conformal Ward Identity. The Ward identity is modified in an important way in the presence of domains with a boundary.
- **Boundary Conformal Field Theory**: I will describe the role of boundary conditions in our two-dimensional CFT. We will see that the holomorphic and antiholomorphic generators

no longer decouple, and are in fact coupled due to boundary conditions known as "**gluing conditions**". Most important for us will be the abovementioned modification of the Conformal Ward Identity, and the Cardy Doubling Trick which permits evaluation of correlation functions in a boundary CFT using techniques already known for the open domain CFT.

- **The 2D Ising Model in the Upper Half Plane**: We will revisit our friend, the 2D Ising Model and try to understand its surface critical behavior through the machinery of 2D Boundary CFT. We will get an explicit form for the asymptotic spin-spin correlation function, from which we will be able to extract the critical exponents for ordinary and extraordinary transitions.
- **The Boundary State Formalism**: There is a more rigorous formalism to study boundary conformal field theories, developed by Cardy and others, which we will discuss in the context of the 2D Ising Model on the upper half plane. The new concepts that become important for us in this formalism are modular invariance, boundary states and boundary condition changing operators. The boundary state formalism (which is also important for D-branes and string scattering amplitudes) is essential for identifying the allowed conformally invariant boundary conditions, and also the boundary operator content. It turns out that there are local scaling fields on the boundary, known as boundary condition changing operators, which identify the transitions between different boundary states.

2 Surface Critical Behavior

2.1 Mean Field Theory for the Ferromagnetic Ising Model

Our motivating example will be the ferromagnetic Ising Model, the partition function for which is

$$Z = \mathrm{Tr}\exp\left[\frac{\beta}{2}\sum_{r,r'} J(r-r')s(r)s(r') + \beta H\sum_r s(r)\right] \tag{1}$$

Here $s(r) = \pm 1$, $J(r-r')$ is the ferromagnetic exchange coupling, and H is a uniform applied magnetic field.

In mean field theory, we approximate the interacting situation by a simpler noninteracting partition function. This is achieved by writing

$$s(r)s(r') = (M + (s(r) - M))(M + (s(r') - M)) \tag{2}$$

and approximating the exponent of (1) by

$$\frac{\beta}{2}\sum_{r,r'} J(r-r')\left(M(s(r)+s(r')) - M^2\right) + \beta H\sum_r s(r) \approx -\frac{1}{2}N\beta J M^2 + \beta(JM+H)\sum_r s(r) \tag{3}$$

whereas

$$J \equiv \sum_{r'} J(r-r') \tag{4}$$

and N is the total number of sites. Mean field theory thus neglects *correlations* between spins on neighboring sites. So we anticipate that **it will yield correct results only when the correlation length is small**.

2.2 Critical Temperature

The partition function can be written as

$$Z \approx e^{-\frac{1}{2}N\beta J M^2}\left[2\cosh\beta(JM+H)\right]^N \tag{5}$$

The free energy per site $f \equiv -(k_B T/N)\ln Z$ for a given M, is

$$f_{MF}(M) = \frac{1}{2}JM^2 - \frac{1}{\beta}\ln\cosh\beta(JM+H) \tag{6}$$

The value of M is the one that minimizes $f_{MF}(M)$. This yields the condition,

$$M = \tanh\beta(JM+H) \tag{7}$$

For $H = 0$ and small M¡ we can Taylor expand $f_{MF}(M)$ as

$$f_{MF} = \text{const.} + \frac{1}{2}J(1-\beta J)M^2 + O(M^4) \tag{8}$$

So, when $T > J/k_B$ the only minimum is $M = 0$. This corresponds to the paramagnetic phase. For $T < J/k_B$, symmetry is spontaneously broken.

For nonzero field ($H \neq 0$), the system has only one minimum, and hence a unique value of M.

The temperature $T_C^{MF} = J/k_B$ is the mean field approximation to the critical temperature of the system. But it is an overestimate, because mean field theory neglects fluctuations which result in disordering and consequent suppression of the true T_C relative to the mean field value.

2.3 Mean Field Theory for the Correlation Function

The spin-spin correlation function, which will be an important object in the CFT story later, is

$$\langle s(r)s(0)\rangle \equiv \frac{\text{Tr } s(r)s(0)e^{\frac{1}{2}\beta\sum_{r',r''}J(r'-r'')s(r')s(r'')}}{\text{Tr } e^{\frac{1}{2}\beta\sum_{r',r''}J(r'-r'')s(r')s(r'')}} \tag{9}$$

The Hamiltonian is symmetric under change of signs of all spins. Hence, if we insert the identity $1 = \delta_{s(0),1} + \delta_{s(0),-1}$, the spin-spin correlation function can be written as

$$\langle s(r)s(0)\rangle \equiv \frac{\mathrm{Tr}' s(r) e^{\frac{1}{2}\beta\sum_{r',r''} J(r'-r'')s(r')s(r'')}}{\mathrm{Tr}' e^{\frac{1}{2}\beta\sum_{r',r''} J(r'-r'')s(r')s(r'')}} \tag{10}$$

where the primes on the trace imply that we keep $s(0) = 1$ fixed, and trace over all the other spins. So this two-point function (also called $G(r)$) equals the response of the system to an applied field which is localized at the origin. In other words, this is a step response of the system.

Repeating the mean-field theory arguments, the self-consistent equation that we get from minimizing the free energy in this case is

$$M(r) = \tanh\left[\beta\left(\sum_{r'} J(r-r')M(r')\right)\right]$$

Our aim is not to solve this equation analytically of course, but to extract its asymptotic (large r behavior), for instance in the high temperature regime when we know $M(r)$ is small. In this regime, we can Taylor expand the right hand side, to get

$$M(r) = \beta\sum_{r'} J(r-r')M(r') + \text{small localized corrections (when } r \text{ is small)} \tag{11}$$

Assuming that the corrections are localized, they can be approximated by a delta function source. Exploiting the convolution on the right hand side, we can solve this equation in the Fourier domain, to get

$$\tilde{M}(k) \approx \frac{\text{const.}}{1-\beta J(1-R^2k^2)}$$

where $\tilde{J}(k) = J(1-R^2k^2) + O(k^4)$, and

$$R^2 = \frac{\sum_r r^2 J(r)}{\sum_r J(r)} \tag{12}$$

is a measure of the squared range of exchange interactions. Finally, the correlation function in mean field theory is determined (in Fourier space) as

$$\tilde{G}(k) \sim \frac{\text{const.}\cdot R^{-2}}{k^2+\xi^{-2}}$$

where $\xi = Rt^{-1/2}$. The position space correlation function has the Ornstein-Zernicke form $G(r) \sim e^{-r/\xi}\frac{1}{r^{(d-1)/2}}$ for $r >> \xi$. For $r << \xi$, we have $e^{-r/\xi}/r^{d-2+\eta}$. As $\xi \propto t^{-1/2}$, the mean field value of the exponent $\nu = 1/2$. At the critical point, $G(r)$ is given by the inverse Fourier transform of $1/k^2$, which for large r goes as $1/r^{d-2}$. So the mean field value of the exponent η is $\eta = 0$.

2.4 Surface Critical Behavior from Mean Field Theory

Our starting point is

$$M(r) = \tanh\left[\beta\left(\sum_{r'} J(r-r')M(r')\right)\right]$$

The mean field transition in the bulk occurs when

$$\beta \sum_{r'} J(r-r') = 1 \tag{13}$$

This should be thought of as the appropriate generalization of $\beta J = 1$.

As there are fewer sites at the boundary, we normally expect (roughly speaking),

$$\sum_{r'} J(r-r')\bigg|_{surface} < \sum_{r'} J(r-r')\bigg|_{bulk} \tag{14}$$

Consequently,

$$M_{surface} < M_{bulk} \tag{15}$$

The phase transition on the boundary in such a case is termed the **ordinary transition**.

Again, close to the bulk critical point, $M(z)$ is small and depends weakly on the distance z from the boundary. So, we can Taylor expand to get

$$\sum_{r'} J(r,r')M(r') = J(z)M(z) + \frac{1}{2}R^2 J \frac{\partial^2 M(z)}{\partial z^2} + \cdots \tag{16}$$

where $J(z) = \sum_{r'} J(r-r')$, and J and R^2 are as defined in the previous section.

Cardy assumes the form

$$J(z) \sim J\left(1 - \frac{R^2}{2\lambda}\delta(z)\right) \tag{17}$$

for the coupling. It is nonzero except in a narrow region near the boundary. The magnetization therefore satisfies an ordinary differential equation of the form†

$$\frac{1}{R^2}\frac{\partial^2 M}{\partial z^2} + M - \frac{R^2}{2\lambda}\delta(z)M = (\beta J)^{-1}\tanh^{-1} M \tag{18}$$

$$= (\beta J)^{-1}\left(M + \frac{1}{3}M^3 + \cdots\right) \tag{19}$$

†There is perhaps a typo in the delta function term in Cardy's book. He is missing a factor of 2 in front of λ.

with the boundary condition

$$M = 0 \text{ for } z < 0 \tag{20}$$

The boundary value problem near the bulk critical point $\beta J = 1$ is

$$\frac{1}{2}R^2\frac{\partial^2 M}{\partial z^2} = tM + \frac{1}{3}M^3 \text{ for } z > 0 \tag{21}$$

where $t = (T - T_C)/T_C$. This has to be supplemented by a boundary condition, which one can get by integrating the original d.e. around $z = 0$. This gives

$$\left.\frac{\partial M}{\partial z}\right|_{0^+} - \left.\frac{\partial M}{\partial z}\right|_{0^-} = \frac{M(0^+) - M(0^-)}{\lambda} \tag{22}$$

Cardy works instead with ‡

$$\frac{\partial M}{\partial z} = \frac{M}{\lambda} \text{ at } z = 0^+ \tag{23}$$

Extrapolated into $z < 0$, $M(z)$ would vanish at $z \approx -\lambda$. This microscopic distance is known as the **extrapolation length**.

- For $T > T_C$, the only solution has $M = 0$.
- For $T < T_C^{bulk}$, $M \to M_{bulk} \propto (-t)^{1/2}$ as $z \to \infty$. This is consistent with the mean field bulk magnetization exponent $\beta = \frac{1}{2}$. M in fact approaches this value exponentially fast, over a length scale given by the *bulk correlation length* $\xi \propto |t|^{-1/2}$.

A generic solution that satisfies these properties is

$$M(z) = (-t)^{1/2} f\left(\frac{z+\lambda}{Rt^{-1/2}}\right) \tag{24}$$

where $f(X \to \infty) = \text{const.}$ and $f(0) = 0$.

The magnetization near $z = 0$ has a different temperature dependence. As f is analytic,

$$f\left(\frac{\lambda}{Rt^{-1/2}}\right) = O(t^{1/2}) \tag{25}$$

So, $M(0) \propto (-t)^{\beta_1}$ where $\beta_1 = 1$ is an example of a **surface exponent**, which differs from the bulk exponent.

‡Presumably the assumption is that the magnetization identically vanishes for $z < 0$ first. The solution is *then* analytically continued into $z < 0$ to get an estimate of the extrapolation length. This should really not be a source of confusion. But if one likes, one can simply assume a general solution with an extrapolation length to begin with.

2.5 Critical behavior from the correlation function: method of images

The correlation function $G(r_1, r_2)$ at the bulk critical point satisfies the Laplace equation,

$$\nabla^2 G = 0 \tag{26}$$

for large $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$, in the mean field approximation. In the bulk, the solution to this equation has the spherically symmetric form $r_{12}^{-(d-2)}$ leading to the (bulk) exponent $\eta = 0$.

However, when a boundary is present, G satisfies the same boundary conditions as M above, i.e. $G = 0$ whenever z_1 or $z_2 = -\lambda$. This problem can be solved using the method of images familiar from electrostatics. The solution is like the electrostatic potential at $\boldsymbol{r}_2 = (\boldsymbol{x}_2, z_2)$ due to a charge at $\boldsymbol{r}_1 = (\boldsymbol{x}_1, z_1)$ in the presence of a conducting boundary at $z = -\lambda$, i.e.

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) \sim \left[\frac{1}{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (z_1 - z_2)^2)^{d/2-1}} - \frac{1}{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (z_1 + z_2 + 2\lambda)^2)^{d/2-1}}\right]$$

- When z_1 and z_2 are much greater than $|\boldsymbol{x}_1 - \boldsymbol{x}_2|$, we get the bulk behavior $r_{12}^{-(d-2)}$.
- If $\boldsymbol{r}_1$ is close to the boundary but $\boldsymbol{r}_2$ is far away, we get $r_{12}^{-(d-1)}$ times some function of the angle that $\boldsymbol{r}_{12}$ makes with the normal to the surface.
- If both points are near or on the boundary, the scaling behavior is r_{12}^{-d}.

So the bulk scaling dimension is $x_h = d/2 - 1$ for the magnetization operator in the bulk, and the boundary scaling dimension is $x_{h,s} = d/2$ for the magnetization operator close to the boundary.

This can be made more precise through RG arguments as Cardy does in his book.

2.6 Extraordinary and special transitions

If the surface couplings are enhanced above their bulk values, the surface can order at a temperature higher than that of the bulk. This refers to a **surface transition**.

As the temperature is further reduced, the bulk orders at the bulk critical temperature (which by assumption, is lower than the surface critical temperature). As surface quantities are coupled to the bulk, they can also exhibit singularities at this point. This is referred to as the **extraordinary transition**.

The surface undergoes two separate transitions as the temperature is lowered. Finally, as the surface enhancement is lowered, these two lines of critical points meet at a multicritical point known as the **special transition**. We will revisit these transitions from a CFT perspective, shortly.

3 Introduction to $2D$ Conformal Field Theory

Conformal field theory is most powerful in $d = 2$, because the algebra of infinitesimal conformal transformations (i.e. those that preserve angles but not necessarily lengths) is finite in $d > 2$ and too restrictive in $d < 2$.

The 2D space in which the CFT lives can have either a Minkowski or an Euclidean metric. We consider the Euclidean case, as it is most relevant to statistical mechanics.

Two dimensional CFTs are

- Massless
- Local
- Relativistic
- Renormalized

quantum field theories. It is convenient to use complex coordinates to study them.

$$z = x + iy \tag{27}$$
$$\bar{z} = x - iy \tag{28}$$

Analytic functions automatically preserve angles, and an infinitesimal conformal transformation has the form

$$z \mapsto z + \epsilon(z) \tag{29}$$

where $\epsilon(z)$ is analytic. There are infinitely many such independent transformations.

> The notation z^* refers to the complex conjugate of z, but $\bar{z}$ is treated as an independent variable (which may later be set equal to z^*). Correlation functions can expressed in terms of holomorphic and antiholomorphic functions, e.g. $\langle\phi(z, \bar{z}\cdots\rangle = \sum f(z)g(\bar{z})$. The "chiral half" of a correlation function refers to its z-dependence (keeping $\bar{z}$ fixed.

The conformal group on the Riemann sphere is finite dimensional and consists only of Möbius transformations,

$$z \mapsto \frac{az + b}{cz + d} \tag{30}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. These transformations form the Lie group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

> In $d \geq 2$, the conformal *group* of global conformal transformations is always finite dimensional. However the conformal *algebra* of infinitesimal conformal transformations is infinite in $d = 2$ and finite in $d > 2$ dimensions.

In general, to the fields $\phi(z,\bar{z})$ we can associate a **scaling dimension** Δ and a **spin** J (denoted in statmech literature as x and s respectively), according to their behavior under transformations of global rescaling and rotation. That is, under the map $w = re^{i\theta}z$ for some fixed r and θ, we have

$$\phi(z,\bar{z}) \longmapsto r^{\Delta}e^{i\theta J}\phi(w,\bar{w}) \tag{31}$$

Fields which transform with $J = 0$ are referred to as **spinless** or **diagonal** fields.

Any conformal transformation $z' = f(z)$ looks locally like a combination of a rescaling and rotation. Primary fields are the fields in the CFT which only see this local behavior, and whose transformation properties depend only on the first derivative of f.

Primary fields are local densities $\phi_j(z,\bar{z})$ that under conformal mappings

$$z \longmapsto z' = f(z) \tag{32}$$

transform as

$$\phi(z,\bar{z}) = \left(\frac{df}{dz}\right)^{h}\left(\frac{d\bar{f}}{d\bar{z}}\right)^{\bar{h}}\phi'(z',\bar{z}') \tag{33}$$

The numbers $(h,\bar{h})$ are called the **conformal weights** of the primary field. It follows that correlators of primary fields transform as

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\cdots\rangle = \prod_j \left(\frac{df(z_j)}{dz_j}\right)^{h_j}\left(\frac{d\bar{f}(\bar{z}_j)}{d\bar{z}_j}\right)^{\bar{h}_j}\langle\phi_1(z_1',\bar{z}_1')\phi_2(z_2',\bar{z}_2')\cdots\rangle \tag{34}$$

A **primary** field ϕ transforms by definition as (33). However, if (33) holds *only* for global conformal transformations, i.e. for $f \in SL(2,\mathbb{C})/\mathbb{Z}_2$, then ϕ is said to be **quasi-primary**.

A primary field is always quasi-primary but the converse is not true.

Not all fields in a CFT are primary or quasi-primary. Such fields are called secondary fields (for example, the derivative of a primary field of conformal dimension $h \neq 0$ is secondary).

The infinitesimal form of the conformal transformation of primary fields is

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z,\bar{z}) \tag{35}$$

The local fields can be normalized such that their 2-point functions have the generic form,

$$\langle\phi_j(z_j,\bar{z}_j)\phi_k(z_k,\bar{z}_k)\rangle = \frac{\delta_{jk}}{(z_j - z_k)^{2h_j}(\bar{z}_j - \bar{z}_k)^{2\bar{h}_j}} \tag{36}$$

These local fields satisfy an algebra known as the **Operator Product Expansion**:

$$\phi_i(z_1, \bar{z}_1) \cdot \phi_j(z_2, \bar{z}_2) = \sum_k C_{ijk}(z_1 - z_2)^{-h_i-h_j+h_k}(\bar{z}_1 - \bar{z}_2)^{-\bar{h}_i-\bar{h}_j+\bar{h}_k}\phi_k(z_1, \bar{z}_1) + \cdots \tag{37}$$

The OPE is valid when it is inserted into higher-order correlation functions, in the limit when $|z_1 - z_2|$ is much less than separations of the other points. The ellipses $(\cdots)$ denote contributions of other non-primary scaling fields.

The **structure constants** C_{ijk} along with the conformal weights, characterize the particular CFT.

An important object for a CFT is the stress tensor, $T^{\mu\nu}(x)$, defined by[§]

$$\delta S = -\frac{1}{2}\int d^2\boldsymbol{x} T^{\mu\nu}\delta g_{\mu\nu} \tag{38}$$

That is, the stress tensor is the response of the action to a local change in metric. The algebra of infinitesimal conformal transformations in $d = 2$ is ∞-dimensional, so there are strong constraints on a CFT. It is possible to study such a theory without having an explicit form of the action. The only information required is the behavior under conformal transformations, which is encoded in the stress tensor.

Symmetry of the theory under translations and rotations $\implies$ $T^{\mu\nu}$ is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$ and symmetric.

Scale invariance $\implies$ $T^{\mu\nu}$ is traceless, i.e. $\Theta = T^\mu{}_\mu = 0$. But quantum corrections can affect the vanishing of $T^\mu{}_\mu$ for a scale invariant classical field theory. In fact, $\Theta \propto \beta(g)$, the RG beta function.

So, a QFT is a CFT only when $\beta = 0$, that is, at an RG fixed point.

For a $d = 2$ CFT, the components

$$T_{z\bar{z}} = T_{\bar{z}z} = 4\Theta \tag{39}$$

vanish, while the conservation equations become

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0 \tag{40}$$

So the correlators of $T(z) \equiv T_{zz}$ are locally analytic (in fact, globally meromorphic) functions of z, whereas correlators of $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$ are analytic functions of $\bar{z}$ (called *anti-analytic* or *anti-holomorphic*). This property of $d = 2$ CFT makes it tractable.

[§]Definitions vary widely, with different factors of 2 or 2π.

3.1 The Conformal Ward Identity on the full Complex Plane

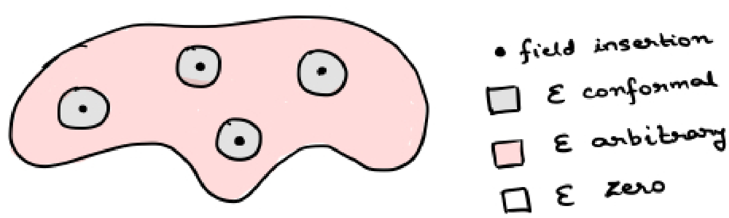
Under an infinitesimal coordinate transformation,

$$\delta S = -\int d^2\boldsymbol{x}\partial_\mu \varepsilon_\nu(\boldsymbol{x})T^{\mu\nu}(\boldsymbol{x}) \tag{41}$$

For a collection of fields X (X is a product of primary fields), the path integral formulation yields

$$\int [d\Phi] X\delta S e^{-S[\Phi]} = \int [d\Phi]\delta X e^{-S[\Phi]} \tag{42}$$

Consider the specific product of n primary fields, $X = \phi_1(z_1, \bar{z}_1)\ldots\phi_n(z_n, \bar{z}_n)$. Consider an $\varepsilon(z)$ that is conformal in a small disc around each z_k and is otherwise arbitrary in a compact region containing all the discs. Outside this region, $\varepsilon(z) = 0$. Let D denote the collection of all n discs. As $\varepsilon(z)$ is a symmetry of the action inside D, we have $\delta S|_D = 0$.



Geometry of the Ward Identity for the CFT on the entire complex plane.

From the infinitesimal form of the conformal transformation (35) and (41), we get

$$\begin{aligned} &-\int_{\mathbb{R}^2-D} d^2x\partial_\mu\varepsilon_\nu\langle T^{\mu\nu}(x)\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle \\ &= \sum_{k=1}^{n}\left[h_k\varepsilon'(z_k) + \varepsilon(z_k)\partial_{z_k} + \bar{h}_k\bar{\varepsilon}'(\bar{z}_k) + \bar{\varepsilon}(\bar{z}_k)\partial_{\bar{z}_k}\right]\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle \end{aligned} \tag{43}$$

We can use Gauss's theorem to reduce the left hand side of (43) to a surface integral, and a term containing $\varepsilon_\nu(x)\partial_\mu T^{\mu\nu}$ integrated over $\mathbb{R}^2 - D$. We can choose an arbitrary $\varepsilon(x)$ in $\mathbb{R}^2 - D$. As the right hand side is independent of the values of ε outside D, this is consistent only if T is conserved.

This derivation, when repeated for a boundary CFT yields (in addition to the conservation of T), an appropriate boundary condition for T.

When the position of the stress tensor does not coincide with insertion points of other fields, we have

$$T_{xy} = T_{yx}; T_{xx} + T_{yy} = 0; \partial_x T_{xx} + \partial_y T_{yx} = 0; \partial_x T_{xy} + \partial_y T_{yy} = 0 \tag{44}$$

These relations arise from invariance under rotations and rescaling, and conservation of the stress tensor.

Using the conservation of $T_{\mu\nu}$, (43) becomes

$$
\begin{aligned}
&-\int_{\partial D} \varepsilon_\mu(x) n_\nu(x) \langle T^{\mu\nu}(x)\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle \\
&= \sum_{k=1}^{n} \left[h_k \varepsilon'(z_k) + \varepsilon(z_k)\partial_{z_k} + \bar{h}_k \bar{\varepsilon}'(\bar{z}_k) + \bar{\varepsilon}(\bar{z}_k)\partial_{\bar{z}_k}\right] \langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle
\end{aligned}
\tag{45}
$$

where $n_\nu(x)$ is an inward pointing vector normal to the boundary of D.

There is more we can do, to get different forms of the Ward identity, including stronger forms (see [5] for examples). But this is not required for BCFT.

Taking care of signs and factors in transforming (45), we finally get

$$
\langle T(\zeta)\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle = \sum_{k=1}^{n} \left[\frac{h}{(\zeta - z_j)^2} + \frac{1}{\zeta - z_j}\frac{\partial}{\partial z_j}\right] \langle T(\zeta)\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle + \text{reg}(\zeta)
\tag{46}
$$

where $\text{reg}(\zeta)$ is a regular function on the complex plane. A similar expression holds for $\bar{T}(\bar{z})$.

4 Boundary Conformal Field Theory

Any statistical/quantum system is characterized by a set of boundary conditions at the surface. For the model to be have a CFT description at criticality, conformal transformations must map the boundary onto itself and preserve the boundary conditions. This places restrictions on the symmetry of the model. Holomorphic and antiholomorphic fields no longer decouple, and only half of the conformal generators remain.

We will restrict a bulk CFT from the full complex plane to the upper half plane (UHP). The Riemann Mapping Theorem (see below) is cited in the literature to justify that studying CFTs on the UHP is not restrictive.

However [7], it may not always be possible to map Killing vectors associated with a smooth but nondifferentiable boundary with those preserved by a smoother boundary (an intuitive example is a wedge in $d = 2$), even if the primary fields themselves can be conformally mapped.

Riemann Mapping Theorem: If U is a non-empty simply connected open subset of the complex plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from U onto the open disk

$$D = \{z \in \mathbb{C} : |z| < 1\} \tag{47}$$

"Proof":

Consider some point z_0 in the given U. We assume U is bounded and its boundary is smooth. Define $f(z) = (z - z_0)e^{g(z)}$ where $g = u + iv$ is some holomorphic function with real part u and imaginary part v. Clearly, $z = z_0$ is the only zero of f. In order to have $|f(z)| = 1$ for all $z \in \partial U$, we need

$$u(z) = -\ln|z - z_0| \tag{48}$$

As $u(z)$ is the real part of a holomorphic function, $u(z)$ is essentially a harmonic function, i.e. it satisfies Laplace's equation. We need a function u defined on all of U with the given boundary condition. The existence of such a function is guaranteed by the Dirichlet principle.

So now, we can use the Cauchy-Riemann equations to find the $v(z)$ corresponding to this $u(z)$ (this requires U to be simply connected). ■

For a CFT defined on the UHP, conformal transformations have two roles:

1. Transformations that change the geometry (e.g. from the UHP to the unit disc) allow us to express correlators in different geometries in terms of those on the UHP.
2. Transformations that map the UHP to itself impose constraints on the correlators on the UHP.

Conformal transformations that map the real axis onto itself are obtained by keeping the Möbius parameters a, b and c real (recall that $ad - bc = 1$, so constraining 3 to be real automatically forces the fourth, i.e. d to be real). So, the global conformal group is half as large as it is for the entire plane.

Infinitesimal local conformal transformations $z \mapsto z + \varepsilon(z)$ will map the real axis onto itself only if $\varepsilon(z) = \bar{\varepsilon}(\bar{z})$ for $z = \bar{z}$, i.e. if $\varepsilon(\bar{z}) = \bar{\varepsilon}(z)$, i.e. ε is real on the real axis. This strong constraint eliminates half of the conformal generators: the holomorphic and antiholomorphic sectors of the theory are no longer independent.

Let us first look at correlators on the UHP. So we consider conformal transformations that map the UHP to itself. We have seen how Ward identities were obtained for the CFT on the full complex plane. We want to repeat the exercise on the UHP. So we consider again a collection of primary fields X and an infinitesimal transformation $\varepsilon(x,y)$ with the following properties:

- $\varepsilon(x,y)$ is a continuous function $UHP \to \mathbb{C}$.
- For all x, we have $\varepsilon_y(x,0) = 0$. So the transformation $(x,y) \mapsto (x,y) + \varepsilon(x,y)$ maps the UHP to itself (because on the x-axis $(x, y = 0) \mapsto (x, y = 0) + (\epsilon_x(x, y = 0), \epsilon_y(x, y = 0) = 0) = (x + \epsilon_x(x, y = 0), 0))$.
- $\varepsilon(x,y)$ is conformal in a semi-disc D containing all the fields X.
- Suppose K is a compact set such that $D \subset K$. Then, $\varepsilon(x,y)$ is arbitrary in $K - D$.
- $\varepsilon(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2 - K$.

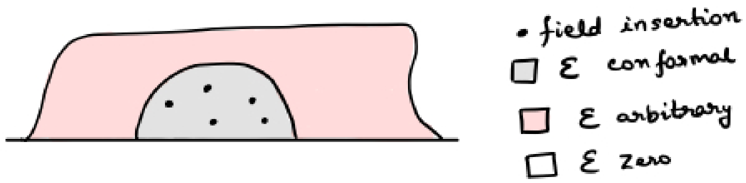


Figure: Geometry of the Ward Identity for the CFT on the Upper Half Plane.

Start from the relation,

$$\int [d\Phi] X \delta S e^{-S[\Phi]} = \int [d\Phi] \delta X e^{-S[\Phi]} \tag{49}$$

for the infinitesimal transformation $\varepsilon(x,y)$. **We demand that the boundary condition along the real line is conformally invariant**. That is, we demand that $\delta S|_D = 0$: the action, including a segment of the boundary, does not change under the conformal transformation.

As before, δX is given by the generalization of $\delta_{\varepsilon,\bar{\varepsilon}}\phi(z,\bar{z}) = ...$ And as before, the right hand side is independent of the values of $\varepsilon(x,y)$ outside D. Integrating by parts, we get

$$\int [d\Phi] \delta X e^{-S[\Phi]} = -\int_{\partial(K-D)} n_\nu(\boldsymbol{x}) \varepsilon_\mu(\boldsymbol{x}) \langle T^{\mu\nu} X \rangle + \int_{K-D} \varepsilon_\mu(\boldsymbol{x}) \partial_\nu \langle T^{\mu\nu}(\boldsymbol{x}) X \rangle \tag{50}$$

where $n_\nu(\boldsymbol{x})$ is an outward pointing normal to the boundary $\partial(K-D)$. From the second term, we conclude that $T^{\mu\nu}$ is conserved away from the insertion points of other fields, $\partial_\mu T^{\mu\nu} = 0$. But we get a new condition from the first term.

On the real line, each portion of the integral of the first term is of the form $\int_a^b \varepsilon_x(x,0) \langle T^{xy}(x,y) X \rangle dx$. For this to be independent of $\varepsilon_x(x,0)$, we must have $T^{xy}(x,0) = 0$.

The condition $T^{xy}(x,0) = 0$ implies no energy flows across the boundary.

In summary, the condition $T^{xy}(x,0) = 0$ is necessary and sufficient to ensure that a boundary condition preserves conformal invariance. In complex coordinates, this takes the form

$$T(z) = \bar{T}(\bar{z}) \text{ for } z = \bar{z} \tag{51}$$

This constraint does not fix the boundary condition uniquely, but merely selects a class of boundary conditions allowed by conformal invariance.

4.1 Ward identities on the UHP

In the bulk, the arguments leading to the Ward identity go through without modification. So, correlators involving T and $\bar{T}$ alone are unambiguously defined.

The functions $z \mapsto T(z, z^*)$ and $z \mapsto \bar{T}(z^*, z)$ are *both* analytic (so any correlator involving these fields is analytic in z). They are also equal on the real line, i.e. $T(x) = \bar{T}(x)$. By analytic continuation, this should hold for all $z \in \mathbb{C}$:

$$T(z, z^*) = \bar{T}(z^*, z) \text{ for all } z \in \mathbb{C} \tag{52}$$

On the UHP, all we can say is that the only poles of $T(\zeta, \zeta^*)$ located in the UHP are at $\zeta = z_k$. We also know that the poles of $\bar{T}(\zeta, \zeta^*)$ are equally located at $\zeta = z_k$. Combining this with the above condition, we can infer that the correlator $\langle T(\zeta, \zeta^*)\phi_1(z_1, z_1^*) \cdots \phi_m(z_m, z_m^*)\rangle_{UHP}$ has poles only at $\zeta = z_k$ and $\zeta = z_k^*$ for all k. Further, it decays as ζ^{-4} for $\zeta \to \infty$ along any direction.

So, the Ward identity on the UHP is

$$\begin{aligned}
&\langle T(\zeta, \zeta^*)\phi_1(z_1, z_1^*) \cdots \phi_m(z_m, \bar{z}_m^*)\rangle_{UHP} \\
&= \left(\sum_{k=1}^{m} \left[\frac{h_j}{(\zeta - z_j)^2} + \frac{1}{\zeta - z_j}\frac{\partial}{\partial z_j} \right] + \sum_{k=1}^{m} \left[\frac{\bar{h}_j}{(\zeta - z_j^*)^2} + \frac{1}{\zeta - z_j^*}\frac{\partial}{\partial \bar{z}_j} \right] \right) \langle \phi_1(z_1, z_1^*) \cdots \phi_m(z_m, \bar{z}_m^*)\rangle_{UHP}
\end{aligned} \tag{53}$$

4.2 The Doubling Trick

Comparing the Ward identities for the full complex plane with those for the UHP, we see that the two correlation functions

$$\begin{aligned}
&\left\langle T(\zeta, \zeta^*)\phi_1(z_1, z_1^*) \ldots \phi_m(z_m, z_m^*)\bar{\phi}(z_1^*, z_1) \ldots \bar{\phi}_m(z_m^*, z_m) \right\rangle_{\mathbb{C}} \\
&\langle T(\zeta, \zeta^*)\phi_1(z_1, z_1^*) \ldots \phi_m(z_m, z_m^*)\rangle_{UHP}
\end{aligned} \tag{54}$$

have the same singularities in ζ. Here, $\bar{\phi}_k(w, \bar{w})$ denotes a primary field with conformal weights $(\bar{h}_k, h_k)$ if the field $\phi_k(w, \bar{w})$ has weights $(h_k, \bar{h}_k)$. This suggests the following:

We can think of the correlator on the UHP to be related to the correlator on the full complex plane by reflecting all the fields at the real axis.

The 4-point function on the full plane can be written in terms of *partial waves*, each of which can be further decomposed as a product of holomorphic and anti-holomorphic *conformal blocks*:

$$G_{34}^{21}(x,\bar{x}) \equiv \lim_{z_1,\bar{z}_1\to\infty} z_1^{2h}\bar{z}_1^{2\bar{h}}\langle\phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(x,\bar{x})\phi_4(0,0)\rangle \tag{55}$$

$$= \sum_p C_{34}^p C_{12}^p \mathscr{F}_{34}^{21}(p|x)\bar{\mathscr{F}}_{34}^{21}(p|\bar{x}) \tag{56}$$

where p sums over conformal families, and $x = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$ is the anharmonic ratio (similar expression for $\bar{x}$ with bars on the r.h.s.) and we have used the invariance of x under global transformations to set $z_1 = \infty, z_2 = 1, z_3 = x, z_4 = 0$ in the four point correlation function $\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle$ which depends continuously on x and $\bar{x}$.

The douubling trick says that the 2-point function on the upper half plane, as a function of (z_1, z_2, z_1^*, z_2^*) is a sum of conformal blocks, each of which is the same as the *holomorphic* conformal blocks $\mathscr{F}_{34}^{21}(p|x)$ occuring in the 4-point function in the full plane (which depend on (z_1, z_2, z_3, z_4) continued to $z_3 = z_1^\star$ and $z_4 = z_2^\star$.)

This is known as Cardy's **doubling trick**¶.

¶This relation extends to correlators with several T and $\bar{T}$ fields as well. In this case, a $\bar{T}$ on the UHP becomes a T on the LHP after reflection. So, $\bar{L}_{-n}$ reflects to L_{-n} and so on. The reflection of T itself is the identity, since it is a $(2,0)$ quasiprimary field.

Notes:

1. If the symmetry algebra $\mathcal{W} \times \mathcal{W}$ contains further chiral generators $W(z)$ and $\bar{W}(\bar{z})$ apart from the stress tensor, we can optionally impose boundary conditions that respect all or part of this symmetry as well. This can be done by prescribing an analytic continuation (as was done for $T(z)$). But there is no reason to demand strict equality at the boundary. That is, a more general gluing condition of the form

$$W(z) = \Omega(\bar{W})(\bar{z}) \text{ for } z = \bar{z} \tag{57}$$

can be employed, where $\Omega : \mathcal{W} \to \mathcal{W}$ is a local automorphism of the chiral symmetry algebra which leaves the stress tensor fixed, i.e. $\Omega T = \Omega$. By local, we mean that Ω acts pointwise, i.e. $(\Omega W)(z) = \Omega(W(z))$, so that it commutes with the mode expansion.

2. Consider a correlation function of n primary fields in a minimal model on the UHP. Dress one of the primary fields, say ϕ_1 with L_{-n}'s so that it becomes a null state. This gives rise to a differential equation on the UHP correlation function. The doubling trick implies that this differential equation is exactly the same as the corresponding equation of the ϕ_1-null state in a full plane correlation function with $2n$ primary fields.

3. To get correlation functions in minimal models, we construct conformal blocks, which are solutions to the null state differential equations. The doubling trick tells us that we can use the same set of functions (in different combinations) to describe both full plane and UHP correlators.

5 Characters, Singular states and Irreps

We briefly introduce some additional aspects of CFTs which are needed here.

5.1 The Hilbert space

The Hilbert space can be generated by acting with $\{L_{-n}\}$ and $\{\bar{L}_{-n}\}$ $(n > 0)$ on highest weight states $|h\rangle$. More generally, a basis for descendant states can be obtained by applying raising operators in all possible ways:

$$L_{-k_1} L_{-k_2} \dots L_{-k_n} |h\rangle, \qquad (1 \leq k_1 \leq \dots \leq k_n) \tag{58}$$

where $h' = h + k_1 + \ldots + k_n = h + N$ and N is called the *level* of the state. This generates what is called a Verma module:

l	$p(l)$	
0	1	$\vert h\rangle$
1	1	$L_{-1}\vert h\rangle$
2	2	$L_{-1}^2\vert h\rangle, L_{-2}\vert h\rangle$
3	3	$L_{-1}^3\vert h\rangle, L_{-1}L_{-2}\vert h\rangle, L_{-3}\vert h\rangle$
4	5	$L_{-1}^4\vert h\rangle, L_{-1}^2L_{-2}\vert h\rangle, L_{-1}L_{-3}\vert h\rangle, L_{-2}^2\vert h\rangle, L_{-4}\vert h\rangle$

Table: Lowest states of a Verma Module

Here $p(N)$ is the number of partitions of the integer N:

$$\frac{1}{\varphi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n \tag{59}$$

The Verma Modules generated by $\{L_n\}$ and $\{\bar{L}_n\}$ are denoted by $V(c,h)$ and $\bar{V}(c,\bar{h})$. The Hilbert space is a direct sum of tensor products of V and $\bar{V}$ over all conformal dimensions of the theory, i.e.

$$\mathcal{H} = \sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h}) \tag{60}$$

There may be a finite or infinite number of terms. Also, there may be several terms with the same conformal dimension.

5.2 Characters

To a Verma Module $V(c,h)$ generated by the Virasoro generators L_{-n} $(n > 0)$ acting on the highest weight state $|h\rangle$, we associate a generating function $\chi_{(c,h)}(\tau)$, called the **character** of the module, defined as

$$\chi_{(c,h)}(\tau) = \text{Tr}\ q^{L_0 - c/24} \qquad (q \equiv e^{2\pi i\tau}) \tag{61}$$

$$= \sum_{n=0}^{\infty} \dim(h+n) q^{n+h-c/24} \tag{62}$$

Now,

$$\frac{1}{\varphi(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n \tag{63}$$

and so the generic Virasoro character can be written as

$$\chi_{(c,h)}(\tau) = \frac{q^{h-c/24}}{\varphi(q)} = \frac{q^{h+(1-c)/24}}{\eta(\tau)} \tag{64}$$

where $\eta(\tau)$ is the Dedekind function,

$$\eta(\tau) \equiv q^{1/24}\varphi(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{65}$$

For now, we simply note that characters can be viewed as conformal blocks of the (non-normalized) zero-point correlation function on the torus, which is the torus partition function (in which the left-right decomposition is manifest). Also, the above expressions for the characters are applicable really only for reducible Verma modules, which we denote by $V(c,h)$.

5.3 Singular Vectors and Reducible Verma Modules

Any state $|\chi\rangle$ – other than the highest weight state – that is annhilated by all L_n's $(n > 0)$ is called a **singular vector** (or **null state** or **null vector**).

A singular state generates its own verma module V_χ included in the original module $V(c,h)$.

Singular vectors are orthogonal to the original Verma module:

$$\langle\chi|L_{-k_1}L_{-k_2}\ldots L_{-k_n}|n\rangle = \langle h|L_{k_n}\ldots L_{k_2}L_{k_1}|\chi\rangle^* = 0 \tag{66}$$

In particular, $\langle\chi|\chi\rangle = 0$.

- All the descendants of $|\chi\rangle$ are orthogonal to the whole Verma module $V(c,h)$.
- All the descendants of $|\chi\rangle$ have zero norm. (The evaluation of their norm yields an expression proportional to $\langle\chi|\chi\rangle$.)

Through the operator-state correspondence, a null state $|\chi\rangle$ is associated with a null field $\chi(z)$. A null filed is both primary (meaning that $(L_n\chi)(z) = 0$ if $n > 0$) **and** secondary since it is a descendant of a primary field ϕ_h of dimension h.

The important thing for us is that each null vector has its own differential equation, which acts as a constraint on the correlators and the operator algebera. The net effect of this is an additional truncation of the operator algebra, yielding a finite set of conformal families which closes under fusion.

I will state some well known results about minimal models of CFTs. Specifically, we associate a null field $\chi(z)$ with the null state $|\chi\rangle$. This is a descendant of the primary field $\phi(z)$ of conformal dimension z but is itself a primary field of dimension $(h+2)$. The explicit expression for this null field is

$$\chi(z) = \phi^{(-2)}(z) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\phi(z) \tag{67}$$

The requirement that this null state be orthogonal to the whole Verma Module is equivalent to sasying that the correlator $\langle\chi(z)X\rangle$ vanishes (X is a string of local fields, $X = \phi_1(z_1)\cdots\phi_N(z_N)$).

This implies the following differential equation for the correlator $\langle\chi(z)X\rangle$:

$$\left\{\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right\}\langle\phi(z)X\rangle = 0 \tag{68}$$

where

$$\mathcal{L}_{-n} = \sum_i \left[\frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}}\partial_{w_i}\right] \tag{69}$$

More explicitly, the differential equation (68) is

$$\boxed{\left\{\sum_{i=1}^{N}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right\}\langle\phi(z)X\rangle = 0} \tag{70}$$

5.4 Irreducible Representations

To construct an irrep. of the Virasoro algebra, we quotent out of $V(c,h)$ the null submodule (i.e. all states that differ only by a state of zero norm).

The irreps so obtained are denoted by $M(c,h)$ so as to not confuse them with the reducible Verma Module $V(c,h)$, and they contain fewer states than the generic Verma Module.

The characters of $M(c,h)$ are **not** given by the simple formula for $\chi_{(c,h)}$ discussed earlier, which holds only for $V(c,h)$.

These irreps. are the building blocks for **minimal models**.

6 Minimal Models and the 2D Ising Model

Minimal models are CFTs characterized by a Hilbert space made of a finite number of representations of the Virasoro algebra (Verma modules). The number of conformal families is therefore finite.

The set comprising of a primary field ϕ and all of its descendants is referred to as a **conformal family**, and is denoted by $[\phi]$. The members of a family transform among thesmelves under a conformal transformation. Hence, the OPE of $T(z)$ with any member of the family will be composed solely of other members of the same family.

Suppose p and p' are coprime integers, with $p > p'$. Then CFTs with central charge and conformal dimensions given by

$$c = 1 - 6\frac{(p-p')^2}{pp'} \tag{71}$$

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \tag{72}$$

with r and s being integers lying in the intervals

$$1 \leq r < p' \text{ and } 1 \leq s < p \tag{73}$$

define minimal models, as they contain a finite number of local fields with well-defined scaling behavior.

The range of (r, s) values are bounded by a rectangular region called the Kac table. The conformal dimensions satisfy an obvious symmetry property, namely,

$$h_{r,s} = h_{p'-r,p-s} \tag{74}$$

Due to this, half the rectangle is redundant. Therefore, there are $(p-1)(p'-1)/2$ distinct fields in the theory. **This minimal model is denoted by $\mathcal{M}(p, p')$ and by convention, we take $p > p'$.**

6.1 Unitary Minimal Models

A unitary CFT has no states of negative norm. A necessary condition for a representation of the Virasoro algebra with weight h to be unitary is $h \geq 0$. Therefore, a unitary CFT contains only primary fields with nonnegative conformal dimensions.

This means that the two-point functions of primary fields (except for the identity operator) fall off with distance, i.e.

$$\langle \phi_{h,\bar{h}}(z, \bar{z})\phi_{h,\bar{h}}(0, 0)\rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}} \tag{75}$$

This does happen for the critical Ising Model and is reasonable: the spin-spin correlation function decreases as the separations of the spins increases.

Note that some physical systems such as polymers in 2D have phases described by nonunitary minimal models. In fact, statistical models of of some systems admit critical continuum descriptions with nonunitary CFTs. So, the unitarity condition is not a physical requirement.

6.2 Finite Size Effects

The leading anomalous behavior of the free energy for finite-size effects is governed by the primary field with the smallest dimension.

Bézout's Lemma: If a and b are nonzero integers and $d = \gcd(a, b)$, then $\exists$ integers x and y such that $ax + by = d$.

In particular, take $a = p, b = -p'$. Then, $\exists$ integers r_0 and s_0 satisfying $pr_0 - p's_0 = 1$.

In particular this means

$$h_{r_0,s_0} = \frac{1 - (p - p')^2}{4pp'} \tag{76}$$

is always negative unless $|p - p'| = 1$ in which case, $h_{r_0,s_0} = h_{1,1} = 0$. This corresponds to the identity operator $\phi_{(r_0,s_0)} = \phi_{(1,1)} = \mathbb{I}$. **Therefore, the leading finite-size effect in the free energy is governed only by the central charge of the theory**.

6.3 The 2D Ising Model

The simplest nontrivial unitary minimal model $\mathcal{M}(4, 3)$ describes the critical Ising Model.

The operators are the identity $\mathbb{I}$, the Ising spin σ (a continuum version of the lattice spin σ_i), and the energy density ε (a continuum version of the interaction energy $\sigma_i\sigma_{i+1}$)¶.

The critical exponents η and ν are defined by the critical behavior of the following correlation functions

$$\langle \sigma_i \sigma_{i+m} \rangle = \frac{1}{|m|^{d-2+\eta}} = \frac{1}{|m|^{\eta}} \text{ for } d = 2 \tag{77}$$

$$\langle \varepsilon_i \varepsilon_{i+m} \rangle = \frac{1}{|m|^{2d-2/\nu}} = \frac{1}{|m|^{4-2/\nu}} \text{ for } d = 2 \tag{78}$$

For the exact solution in $d = 2$,

$$\langle \sigma_i \sigma_{i+m} \rangle = \frac{1}{|m|^{1/4}}$$

so $\eta = 1/4$ and

$$\langle \varepsilon_i \varepsilon_{i+m} \rangle = \frac{1}{|m|^2}$$

so $\nu = 1$.

¶The energy density is called a thermal operator, as it couples to the inverse temperature β in the partition function.

Assuming that the scaling fields have no spin, i.e. $h = \bar{h}$, their conformal dimensions are[♫]

$$(h,\bar{h})_\sigma = \left(\frac{1}{16},\frac{1}{16}\right), \qquad (h,\bar{h})_\varepsilon = \left(\frac{1}{2},\frac{1}{2}\right) \tag{79}$$

This can be identified with the minimal model $\mathcal{M}(4,3)$ with central charge $c = 1/2$ (using $p = 4, p' = 3$ to get $h_{r,s} = ((4r-3s)^2-1)/48$). The relevant conformal dimensions are

$$\begin{aligned} h_{1,2} &= \frac{(4-6)^2-1}{48} = \frac{3}{48} = \frac{1}{16}, \qquad \text{corresponds to } (h,\bar{h})_\sigma = \left(\frac{1}{16},\frac{1}{16}\right) \\ h_{2,1} &= \frac{(8-3)^2-1}{48} = \frac{24}{48} = \frac{1}{2}, \qquad \text{corresponds to } (h,\bar{h})_\varepsilon = \left(\frac{1}{2},\frac{1}{2}\right) \end{aligned}$$

So the 3 fields that make up the holmorphic part of the theory have conformal dimensions $0, 1/16$ and $1/2$.

The operator-field correspondence is

$$\begin{array}{cclc} \mathbb{I} & \Longleftrightarrow & \phi_{(1,1)} \text{ or } \phi_{(2,3)} & h_{1,1} = h_{2,3} = 0 \\ \sigma & \Longleftrightarrow & \phi_{(2,2)} \text{ or } \phi_{(1,2)} & h_{2,2} = h_{1,2} = \frac{1}{16} \\ \varepsilon & \Longleftrightarrow & \phi_{(2,1)} \text{ or } \phi_{(1,3)} & h_{2,1} = h_{1,3} = \frac{1}{2} \end{array}$$

[♫]We have $2h_\sigma = 2\bar{h}_\sigma = (1/4)/2$ and $2h_\varepsilon = 2\bar{h}_\varepsilon = 2/2 = 1$.

7 Boundary Conditions for the Critical Ising Model

The conformal transformation law for a primary field is multiplicative, and so preserves homogeneous boundary conditions. The three boundary conditions of interest to us are:

- $\phi|_{\mathbb{R}} = 0$: The Dirichlet boundary condition, responsible for the **ordinary transition**.
- $\phi|_{\mathbb{R}} = \infty$: responsible for the **extraordinary transition**.
- $\left.\frac{\partial\phi}{\partial n}\right|_{\mathbb{R}} = 0$: The Neumann boundary condition, responsible for the **special transition**.

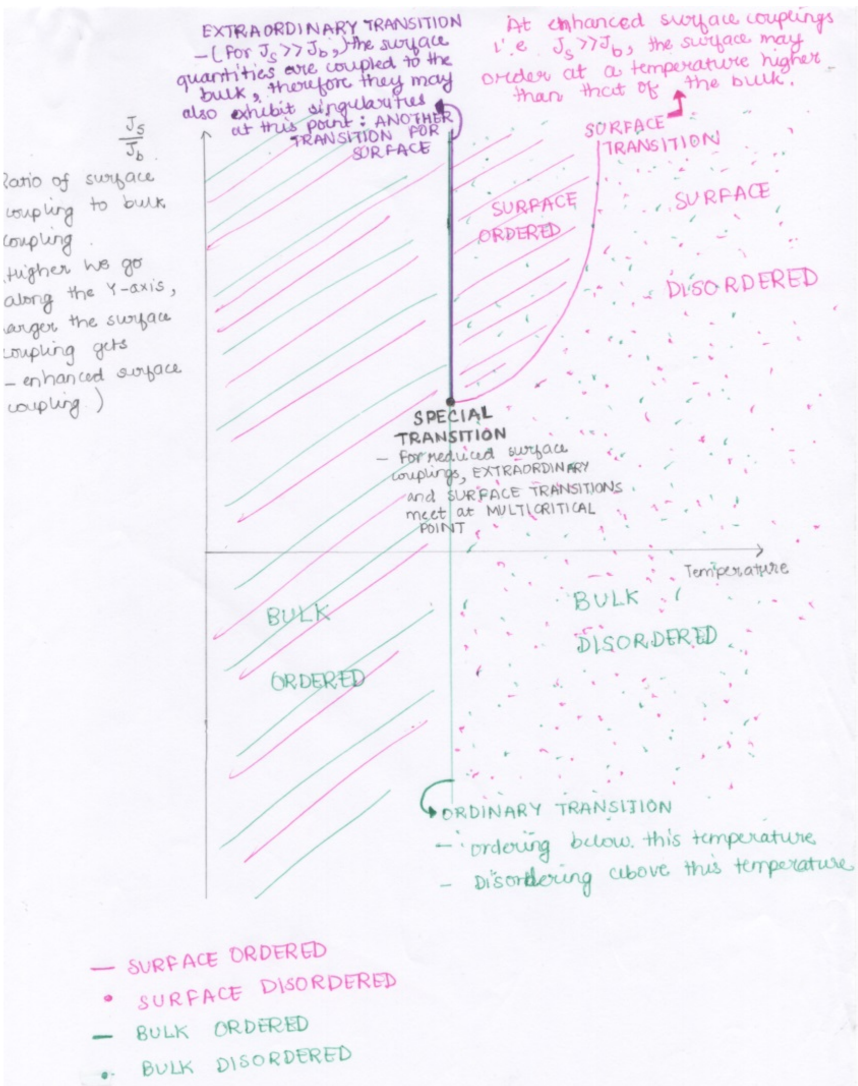


Figure: Phase diagram.

In the context of the Ising Model (the example to follow), ϕ should be thought of as the Landau-Ginzburg field (which corresponds to the bulk spin operator σ in the CFT description). Recall that the Ising Model has $\mathbb{Z}_2$ spin symmetry.

1. **The ordinary transition**
   - ⋆ Associated with the Dirichlet boundary condition, $\phi|_{\mathbb{R}} = 0$.
   - ⋆ Sometimes called the "free" boundary condition, as vanishing of the order parameter on the boundary expresses the absence of constraints on microscopic degrees of freedom.
   - ⋆ The boundary spectrum of the BCFT associated with an ordinary transition has a single relevant scalar operator. This operator is $\mathbb{Z}_2$ odd, and corresponds to $\partial_d \phi$ in the Landau-Ginzburg description.
   - ⋆ The BCFT associated with the ordinary transition preserves $\mathbb{Z}_2$ symmetry (which is a good quantum number for boundary operators).
2. **The extraordinary transition**
   - ⋆ Associated with the boundary condition $\phi|_{\mathbb{R}} = \infty$.
   - ⋆ The surface orders before the bulk due to stronger interactions at the boundary (by "before" we mean $T_C^{surface} > T_C^{bulk}$).
   - ⋆ The order parameter is singular at the boundary (realistically, the divergence is cut off at a microscopic distance from the boundary).
   - ⋆ The extraordinary transition cannot be described in free-field theory. The one-point function does not satisfy the free equations of motion. (But it does appear at first order in the Wilson-Fisher fixed point in $4 - \epsilon$ dimensions.)
   - ⋆ The BCFT associated with the extraordinary transition is said to be the most "stable" as it has no relevant boundary scalar operators.
   - ⋆ It is believed [6] that the lowest-dimension boundary scalar is the so called "displacement operator" T_{dd} (the boundary limit of the stress tensor with both indices pointing in the direction normal to the boundary). T_{dd} has protected conformal dimension $= d$, and hence becomes an irrelevant operator on the $(d-1)$-dimensional boundary.
   - ⋆ The extraordinary transition disappears for $d = 4$.
3. **The special transition**
   - ⋆ Associated with the Neumann boundary condition $\left.\frac{\partial \phi}{\partial n}\right|_{\mathbb{R}} = 0$.
   - ⋆ The BCFT associated with the special transition has two relevant operators: one $\mathbb{Z}_2$ odd and the other $\mathbb{Z}_2$ even, corresponding to ϕ and ϕ^2.
   - ⋆ The BCFT (in $d > 2$) associated with the special transition preserves $\mathbb{Z}_2$ symmetry (which is therefore a good quantum number for boundary operators) just as in case of the ordinary transition.
   - ⋆ There is **no** 2D BCFT associated with the special transition, as the 1D boundary cannot order dynamically at nonzero T. So the surface transition is absent in $d = 2$.

8 2D Ising Model on the Upper Half Plane

The correlation function of interest is

$$G_s(y_1, y_2, \rho) = \langle \sigma(z_1, \bar{z}_1)\sigma(z_2, \bar{z}_2)\rangle \tag{80}$$

Here y_1, y_2 denote the distances of the two points from the real axis, and ρ denotes the "horizontal" distance between the two points. We have argued above that this two point function on the UHP satisfies the same differential equation as the four point function on the entire complex plane. Note that this is different than saying that the two point function on the UHP is equal to the four point function on the entire complex plane, as is deceptively stated in the book by di Francesco et. al. [3]. The equality stated in the book is to be understood at the level of conformal blocks, as clarified earlier in these notes[†].

The differential equation obeyed by the 4-spin correlation function is

$$\left\{\sum_{i=1}^{3}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{1/16}{(z-z_i)^2}\right] - \frac{4}{3}\frac{\partial^2}{\partial z^2}\right\}\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = 0 \tag{81}$$

Recall that the primary field σ has conformal dimension $h_{1,2} = 1/16$ (the Ising Model corresponds to $m = 3$, $r = 1$, $s = 2$ in the terminology introduced above for minimal models). Now, by conformal invariance, the 4-point function is of the form

$$\langle\phi(x_1)\cdots\phi(x_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{2h} F(x) \tag{82}$$

where F is some function of the anharmonic ratio x,

$$x \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{83}$$

Here $z_{ij} \equiv z_i - z_j$ and $z_4 = z$. For us, $h = 1/16$, so

$$\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{1/8} F(x) \tag{84}$$

Substituting this form into the differential equation and performing some change of variables to express everything in terms of x, we get

$$\left[x(1-x)\frac{d^2}{dx^2} + \left(\frac{1}{2} - x\right)\frac{d}{dx} + \frac{1}{16}\right]F(x) = 0 \tag{85}$$

[†]I acknowledge a discussion with Prof. John Cardy, which clarified this issue. It seems that the "doubling trick" is often misquoted in the literature.

This is a particular kind of hypergeometric equation, and it becomes simple with a final change of variable $x = \sin^2\theta$:

$$\left[\frac{d^2}{d\theta^2} + \frac{1}{4}\right] F(\theta) = 0 \tag{86}$$

The general solution to this equation is

$$F(\theta) = c_1 \cos\frac{\theta}{2} + c_2 \sin\frac{\theta}{2} \tag{87}$$

or, expressed back in terms of x,

$$F(x) = c_1\sqrt{1+\sqrt{1-x}} + c_2\sqrt{1-\sqrt{1-x}} \tag{88}$$

The coefficients of the linear combination have to be chosen to satisfy the boundary conditions, which are in turn fixed by the asymptotic behavior of the spin-spin correlation function near the boundary (the real axis). So what are the boundary conditions?

For an ordinary transition, the surface is disordered so $G_s(y_1, y_2, \rho) \to 0$ as $\rho \to \infty$ for fixed y_1 and y_2 (which corresponds to $x \to -\infty$). This fixes $c_2 = -c_1$ and of course, we can set the overall scale $c_1 = 1$ without loss of generality.

For an extraordinary transition, the surface orders before the bulk, which means that as $\rho \to \infty$ for fixed y_1 and y_2 (which is $x \to -\infty$) we have

$$G_s(y_1, y_2, \rho) \sim \langle\sigma(z_1, \bar{z}_1)\rangle_{UHP}\langle\sigma(z_2, \bar{z}_2)\rangle_{UHP} \propto \frac{1}{(y_1 y_2)^{1/8}} \tag{89}$$

This means that for an extraordinary transition $c_2 = c_1$ and of course we can again set $c_1 = 1$. To summarize,

$$F(x) = \begin{cases} \sqrt{\sqrt{1-x}+1} - \sqrt{\sqrt{1-x}-1} & \text{, for Ordinary Transition} \\ \sqrt{\sqrt{1-x}+1} + \sqrt{\sqrt{1-x}-1} & \text{, for Extraordinary Transition} \end{cases} \tag{90}$$

It is conventional to define[‡]

$$\tau \equiv \frac{\rho^2 + (y_1 + y_2)^2}{\rho^2 + (y_1 - y_2)^2} \tag{91}$$

[‡]The reader should be reminded of boundary value problems in two dimensional electromagnetic theory, where conformal invariance results in very similar mathematical expressions.

in terms of which the 4-point functions can be written as

$$G_s(y_1, y_2, \rho) \propto \frac{1}{(y_1 y_2)^{1/8}} \sqrt{\tau^{1/4} \mp \tau^{-1/4}} \tag{92}$$

For fixed finite y_1, y_2 and $\rho \to \infty$, the asymptotic behavior of the correlation function is given by a critical exponent $\eta_{\|}$, i.e.

$$G_s(y_1, y_2, \rho) \sim \frac{1}{\rho^{\eta_{\|}}} \text{ for } \rho >> y_1, y_2 \tag{93}$$

where

$$\eta_{\|} = \begin{cases} 1 & \text{, ordinary} \\ 4 & \text{, extraordinary} \end{cases} \tag{94}$$

9 Fusion Rules and Modular Invariance

9.1 Fusion Rules

We have seen that the full Hilbert space of the CFT can be written as

$$\mathcal{H} = \bigoplus_{h,\bar{h}} n_{h,\bar{h}} \mathcal{M}_h \otimes \bar{\mathcal{M}}_{\bar{h}} \tag{95}$$

The non-negative integers $n_{h,\bar{h}}$ specify the number of distinct primary fields of weights $(h, \bar{h})$ in the CFT.

The consistency of the OPE and the existence of null vectors leads to the fusion algebra of the CFT, which (applying separately to the holomorphic and antiholomorphic sectors) tells us the number of copies of $\mathcal{M}_c$ that occur in the fusion of $\mathcal{M}_a$ and $\mathcal{M}_b$:

$$\mathcal{V}_a \bigodot \mathcal{V}_b = \sum_c N_{ab}^c \mathcal{V}_c \tag{96}$$

where N_{ab}^c are non-negative integers.

For the minimal models $\mathcal{M}(p, p')$ with central charge c and conformal dimensions $h_{r,s}(p, p')$, the fusion algebra closes with a finite number of terms (since $1 \leq r \leq p' - 1$ and $1 \leq s \leq p - 1$). For these models, the fusion algebra takes the form

$$\mathcal{M}_{r_1,s_1} \bigodot \mathcal{M}_{r_2,s_2} = \sum_{r=|r_1-r_2|}^{r_1+r_2-1\,'} \sum_{s=|s_1-s_2|}^{s_1+s_2-1\,'} \mathcal{M}_{r,s} \tag{97}$$

where the primes on the sums indicate that they are to be restricted to the allowed intervals of r and s. More explicitly at the level of fields,

$$\phi_{(r,s)} \times \phi_{(m,n)} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1\, mod\, 2}}^{k_{max}} \sum_{\substack{l=1+|s-n| \\ l+s+n=1\, mod\, 2}}^{l_{max}} \phi_{(k,l)} \tag{98}$$

where

$$k_{max} = \min(r+m-1, 2p'-1-r-m) \tag{99}$$
$$l_{max} = \min(s+n-1, 2p-1-s-n) \tag{100}$$

There is an important theorem according to which the only *unitary* CFTs with $c < 1$ are the minimal models with $p/q = (m+1)/m$, where m is an integer ≥ 3.

9.2 Modular Invariance

The fusion rules limit the values $(h, \bar{h})$ that might appear in a consistent CFT, but do not tell us which ones actually occur, i.e. the values of $n_{h,\bar{h}}$. This is answered by the requirement of modular invariance on the torus.

Consider the theory on an infinitely long cylinder of unit circumference. This is related to the (punctured) plane by the mapping

$$z \mapsto \frac{1}{2\pi} \ln z \equiv t + ix \tag{101}$$

The generator of infinitesimal time translations is related to that for dilatations in the plane by:

$$\hat{H} = 2\pi\left(\hat{D} - \frac{c}{12}\right) = 2\pi(\hat{L}_0 + \hat{\bar{L}}_0) - \frac{\pi c}{6} \tag{102}$$

The last term comes from the Schwarzian derivative. The generator of translations in x is the total momentum operator $\hat{P} = 2\pi(\hat{L}_0 - \hat{\bar{L}}_0)$.

This box is included for the sake of completeness, and is based entirely on a subsection in [3]. Under a conformal transformation $z \mapsto w = f(z)$,

$$T_{new}(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w; z\}\right] \tag{103}$$

where $\{w; z\}$ is the Schwarzian derivative,

$$\{w; z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2 \tag{104}$$

In particular, for $w = \frac{L}{2\pi}\ln z$, $dw/dz = L/(2\pi z)$ and the Schwarzian derivative $= 1/(2z^2)$. So, the energy-momentum tensor on the cylinder is related to that on the plane by

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2\left[T_{pl}(z) - \frac{c}{24}\right] \tag{105}$$

Under a variation of the metric, the free energy varies as

$$\delta F = -\frac{1}{2}d^2x\sqrt{g}\delta g_{\mu\nu}\langle T^{\mu\nu}\rangle \tag{106}$$

On the cylindrical geometry, if we apply an infintesimal scaling of the circumference $L \to (1+\epsilon)L$ or $\delta L = \epsilon L$, this is equivalent to applying a coordinate transformation $w^0 \to (1+\epsilon)w^0$ where w^0 is the coordinate running across the cylinder ($w = w^0 + iw^1$). The coordinate varies according to $\epsilon^\mu = \epsilon w^0\delta^{\mu,0}$ and so $\delta g_{\mu\nu} = -2\epsilon\delta_{\mu,0}\delta_{\nu,0}$. Now,

$$\langle T^{00}\rangle = \langle T_{zz}\rangle + \langle T_{\bar{z}\bar{z}}\rangle = -\frac{1}{\pi}\langle T\rangle = \frac{\pi c}{6L^2} \tag{107}$$

so, the variation of the free energy is

$$\delta F = \int dw^0 dw^1 \frac{\pi c}{6L^2}\frac{\delta L}{L} \tag{108}$$

In general, if there is a free energy per unit area f_0 in the $L \to \infty$ limit, we have

$$\delta F = \int dw^0 dw^1 \left(f_0 + \frac{\pi c}{6L^2}\right)\frac{\delta L}{L} \tag{109}$$

Integrating over w^0 gives a factor of L, and so defining a free energy per unit length F_L (per unit length of the cylinder),

$$\delta F_L = \left(f_0 + \frac{\pi}{c}6L^2\right)\delta L \tag{110}$$

On integration,

$$F_L = f_0 L - \frac{\pi c}{6L} \tag{111}$$

In the main text, we took $L = 1$ (unit circumference), so the addition to the Hamiltonian is $-\pi c/6$.

Up to scale transformations, a general torus is a parallelogram with vertices $(0, 1, \tau, 1+\tau)$ in the complex plane, with the opposite edges identified. [We can make it by taking a cylinder of unit circumference and length $Im(\tau)$ and twisting the ends by a relative amount $Re(\tau)$ and sewing them together.]

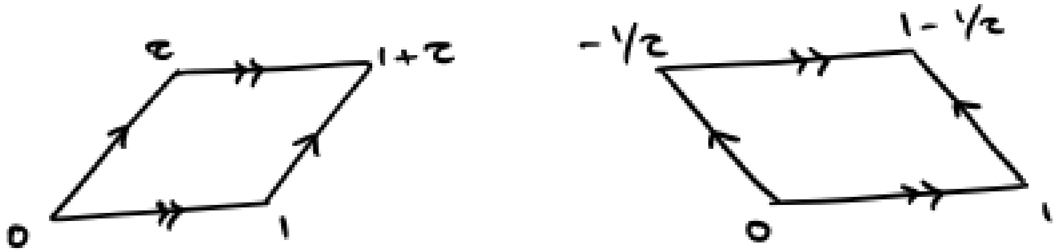


Figure: Two equivalent parametrizations of the torus.

A torus in general can be defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. On the complex plane, these lattice vectors can be denoted by ω_1 and ω_2, which are called *periods* of the lattice. Since the properties of the CFT on the torus do not depend on the overall scale of the lattice or on the absolute orientation of the lattice vectors, the relevant parameter is the ratio $\tau = \omega_2/\omega_1$, called the **modular parameter**.

We take space to lie along the real axis, and time to lie along the imaginary axis. The operator that translates the system parallel to the period ω_2 over a distance a in Euclidean spacetime is

$$\exp - \frac{a}{|\omega_2|}\{H\, Im(\omega_2) - iP\, Re(\omega_2)\} \tag{112}$$

For a being the lattice spacing, this translation takes us from one row of a lattice to the next, but parallel to the period ω_2. If the complete period contains m lattice spacings ($|\omega_2| = ma$) then the partition function is obtained by taking the trace of the above translation operator to the m^{th} power.

$$Z(\omega_1, \omega_2) = \text{Tr } \exp -\{H\, Im(\omega_2) - iP\, Re(\omega_2)\} \tag{113}$$

We have $H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$ and $P = \frac{2\pi i}{L}(L_0 - \bar{L}_0)$. We take ω_1 to be real and equal to L. So, finally,

$$Z(\tau, \bar{\tau}) = \text{Tr } \exp 2\pi i[\tau(L_0 - c/24) - \bar{\tau}(\bar{L}_0 - c/24)] \tag{114}$$

$$= \text{Tr } q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24} \tag{115}$$

where

$$q \equiv e^{2\pi i\tau}, \qquad \bar{q} \equiv e^{-2\pi i\bar{\tau}} \tag{116}$$

So, in terms of characters, the partition function is

$$Z(\tau, \bar{\tau}) = \sum_{h,\bar{h}} n_{h,\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) \tag{117}$$

The transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ give the same torus. Together, these transformations generate the modular group $SL(2, \mathbb{Z})$.

The partition function should be invariant under $SL(2, \mathbb{Z})$ transformations. T-invariance tells us that $(h - \bar{h})$ is an integer, whereas S-invariance places nontrivial constraints on the $n_{h,\bar{h}}$. In fact, the reason why S-invariance is satisfied is due to the remarkable property that characters transform linearly under S:

$$\chi_h(e^{-2\pi i/\tau}) = \sum_{h'} S^{h'}{}_h \chi_{h'}(e^{2\pi i\tau}) \tag{118}$$

For minimal models, the representation is finite dimensional, and the matrix S is symmetric and orthogonal. So, one can get a modular invariant partition function through the diagonal sum

$$Z = \sum_h \chi_h(q) \chi_h(\bar{q}) \tag{119}$$

so that $n_{h,\bar{h}} = \delta_{h,\bar{h}}$.

Let us now consider the partition function on an annulus. This helps classify the allowed boundary conuditions, and boundary operator content. Consider a rectangle of length 1 and height δ with the top and bottom edges identified. This yields an annulus, and we label the two free edges 'a' and 'b'.

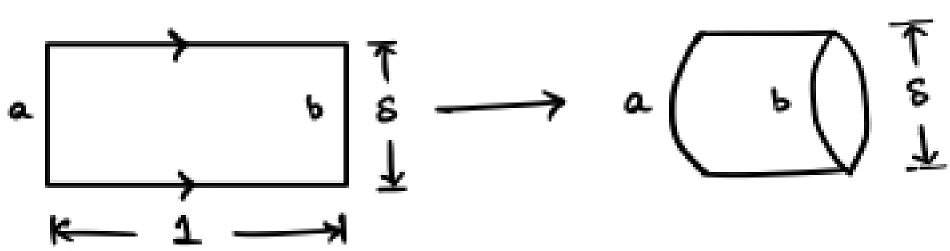


Figure: The annulus (cylinder) constructed using a strip.

The boundary conditions on 'a' and 'b' can in general be different. To compute the partition function $Z_{ab}(\delta)$ we can consider the CFT on an infinitely long strip of unit width. This strip is conformally related to the upper half plane (with the insertion of a boundary condition changing operator at 0 and ∞ if $a \neq b$) by the mapping

$$z \mapsto \frac{1}{\pi} \ln z \tag{120}$$

The generator of infinitesimal transformations along the strip is

$$\hat{H}_{ab} = \pi \hat{D} - \frac{\pi c}{24} = \pi \hat{L}_0 - \frac{\pi c}{24} \tag{121}$$

So the partition function for the annulus is given by

$$Z_{ab}(\delta) = \text{Tr} e^{-\delta \hat{H}_{ab}} \tag{122}$$

$$= \text{Tr} q^{\hat{L}_0 - \pi c/24} \qquad \text{with } q \equiv e^{-\delta\pi} \tag{123}$$

This can be decomposed in terms of characters as

$$Z_{ab}(\delta) = \sum_h n_h^{ab} \chi_h(q) \tag{124}$$

Note that this expression is linear in the characters. The non-negative integers n_h^{ab} indicate the operator content with boundary conditions on the a, b. The lowest value of h with $n_{ab}^h > 0$ corresponds to the conformal weight of the boundary condition changing operator. Other values of h correspond to the other allowed primary fields that may sit at this location.

There is a dual way to study this problem. Specifically, the annulus partition function can be understood as the path integral for the CFT on a circle of unit circumference propagated for an *imaginary time* $1/\delta$. So, Z_{ab} is a matrix element of $e^{-\hat{H}/\delta}$ between **boundary states**, rather than a trace:

$$Z_{ab}(\delta) = \langle a | e^{-\hat{H}/\delta} | b \rangle \tag{125}$$

So the natural question to ask is: how does one characterize boundary states?

The conformal boundary condition applied to the circle is $L_n = \bar{L}_{-n}$. Therefore, we assert that any boundary state $|B\rangle$ lies in the subspace of the Hilbert space that satisfies

$$\hat{L}_n |B\rangle = \hat{\bar{L}}_{-n} |B\rangle \tag{126}$$

Since $\mathcal{H} = \oplus_{h,\bar{h}} n_{h,\bar{h}} \mathcal{M}_h \otimes \bar{\mathcal{M}}_{\bar{h}}$, therefore $|B\rangle$ is also a linear combination of states from $\mathcal{M}_h \otimes \bar{\mathcal{M}}_{\bar{h}}$. Take $n = 0$ in the above relation. This constrains $h = \bar{h}$, so we consider only diagonal CFTs with

$$n_{h,\bar{h}} = \delta_{h,\bar{h}} \tag{127}$$

Let $|h, N; j\rangle$ with $1 \leq j \leq d_h(N)$ be an orthogonal basis for $\mathcal{M}_h$, and $|\overline{h, N; j}\rangle$ be one for $\bar{\mathcal{M}}_h$. The solution to eq. (126) in this subspace is

$$|h\rangle\rangle = \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} |h, N; j\rangle \otimes |\overline{h, N; j}\rangle \tag{128}$$

These are called **Ishibashi States**. The matrix elements of the translation operator along the cylinder between these states can be evaluated as follows

$$\begin{aligned}\langle\langle h'|e^{-\hat{H}/\delta}|h\rangle\rangle &= \sum_{N'=0}^{\infty} \sum_{j'=1}^{d_{h'}(N')} \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} \langle h', N'; j'| \otimes \langle\overline{h', N'; j'}|e^{-(2\pi/\delta)(\hat{L}_0+\hat{\bar{L}}_0-c/12)}|h, N; j\rangle \otimes |\overline{h, N; j}\rangle \\ &= \delta_{h',h} \sum_{N=0}^{\infty} \sum_{j=1}^{d_h(N)} e^{-4\pi/\delta(h+N-c/24)} \qquad (129) \\ &= \delta_{h',h} \chi_h(e^{-4\pi/\delta}) \qquad (130)\end{aligned}$$

Therefore, the characters appearing here are related to those in $Z_{ab} = \sum_h n_h^{ab} \chi_h(q)$ by a modular transformation S. The physical boundary states which satisfy

$$Z_{ab}(\delta) = \sum_h n_h^{ab} \chi_h(q) \tag{131}$$

are called **Cardy States**. They are linear combinations of Ishibashi states:

$$|a\rangle = \sum_h \langle\langle h|a\rangle |h\rangle\rangle \tag{132}$$

Equating the two expressions for Z_{ab}, we get

$$\sum_h n_h^{ab} \chi_h(q) = \langle a|e^{-\hat{H}/\delta}|b\rangle \tag{133}$$

The modular transformation law of χ under S is

$$\chi_h(e^{-2\pi i/\tau}) = \sum_{h'} S_h{}^{h'} \chi_{h'}(e^{2\pi i \tau}) \tag{134}$$

This gives rise the following equivalent conditions

$$n_{ab}^h = \sum_{h'} S^h{}_{h'} \langle a|h'\rangle\rangle \langle\langle h'|b\rangle \tag{135}$$

$$\langle a|h'\rangle\rangle \langle\langle h'|b\rangle = \sum_h S_h{}^{h'} n_{ab}^h \tag{136}$$

known as **Cardy Conditions**. The first Cardy condition states that the right hand side of eq. (135) should give a non-negative integer, and the second one states that the right hand side of eq. (136) should factorize in a and b. These conditions impose very nontrivial constraints on the allowed boundary states and their operator content.

For a diagonal CFT, a complete solution to the Cardy conditions can be found. First of all, it can be shown that elements of the form S_0^h of the matrix S are nonnegative. This allows us to define a boundary state

$$|\tilde{O}\rangle = \sum_h (S_0^h)^{1/2}|h\rangle\rangle, \qquad \text{with } \langle\langle h|\tilde{O}\rangle = \sqrt{S_0^h} \tag{137}$$

and a corresponding boundary condition such that

$$n_{00}^h = \delta_{h,0} \tag{138}$$

For each $h' \neq 0$, we can then define a boundary state

$$\langle\langle h|\tilde{h}'\rangle = \frac{S_{h'}{}^h}{\sqrt{S_0^h}} \tag{139}$$

From the first Cardy condition, we get $n_{h',0}^h = \delta_{h',h}$. So for each allowed h' in the torus partition function, there exists a boundary state $|\tilde{h}'\rangle$ which satisfies the Cardy conditions.

The other requirement is that

$$n_{h',h''}^h = \sum_l \frac{S_l{}^h S_{h'}{}^l S_{h''}{}^l}{S_0^l} \tag{140}$$

should be a non-negative integer. It turns out that the right hand side is precisely what appears in the Verlinde formula that follows from the consistency of the CFT on the torus. The Verlinde formula tells us that the right hand side is equal to the fusion algebra coefficient $N_{h',h''}^h$. As these are non-negative integers, it follows that so are $n_{h',h''}^h$.

Based on these arguments, we infer that for the diagonal CFTs, there is a bijection between the allowed primary fields t the bulk CFT and the allowed conformally invariant boundary conditions. As the minimal models have a finite number of such primary fields, this correspondence can be traced explicitly.

10 Boundary State Formalism for the 2D Ising Model

For minimal models, the modular matrix has a nice explicit form with elements given by

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma}\sin\left(\pi\frac{p}{p'}r\rho\right)\sin\left(\pi\frac{p'}{p}s\sigma\right) \tag{141}$$

Here ρ, σ are the matrix indices and (r, s) are of course fixed for a particular minimal model under consideration. As a matrix, $S^2 = 1$.

For the 2D Ising Model $\mathcal{M}(4, 3)$, the primary fields are

1. $\mathbb{I}$: the identity operator (weight 0),
2. ε: the energy field (weight 1/2),
3. σ: the spin field (weight 1/16),

and the modular matrix S is given by

$$S = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \tag{142}$$

The number of possible conformally invariant boundary conditions equals the number of admissible boundary states. The three possible boundary conditions are: fix the boundary spins at +1, −1 or let them be free. The boundary states are

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle + \frac{1}{2^{1/4}}|\sigma\rangle \tag{143}$$

$$= \frac{1}{\sqrt{2}}|0\rangle\rangle + \frac{1}{\sqrt{2}}\left|\frac{1}{2}\right\rangle\rangle + \frac{1}{2^{1/4}}\left|\frac{1}{16}\right\rangle\rangle \tag{144}$$

$$\left|\frac{\tilde{1}}{2}\right\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle - \frac{1}{2^{1/4}}|\sigma\rangle \tag{145}$$

$$= \frac{1}{\sqrt{2}}|0\rangle\rangle + \frac{1}{\sqrt{2}}\left|\frac{1}{2}\right\rangle\rangle - \frac{1}{2^{1/4}}\left|\frac{1}{16}\right\rangle\rangle \tag{146}$$

$$\left|\frac{\tilde{1}}{16}\right\rangle = |0\rangle - |\varepsilon\rangle \tag{147}$$

$$= |0\rangle\rangle - \left|\frac{1}{2}\right\rangle\rangle \tag{148}$$

The first two states differ in the sign of the state associated with the odd operator σ. Therefore they correspond to the two types of fixed boundary conditions. The third state represents free boundary conditions.

The nontrivial part of the fusion algebra is

$$\mathcal{M}_{1/16} \odot \mathcal{M}_{1/16} = \mathcal{M}_0 + \mathcal{M}_{1/2}, \qquad (\sigma \times \sigma = \mathbb{I} + \varepsilon) \tag{149}$$
$$\mathcal{M}_{1/16} \odot \mathcal{M}_{1/2} = \mathcal{M}_{1/16}, \qquad (\sigma \times \varepsilon = \sigma) \tag{150}$$
$$\mathcal{M}_{1/2} \odot \mathcal{M}_{1/2} = \mathcal{M}_0, \qquad (\varepsilon \times \varepsilon = \mathbb{I}) \tag{151}$$

So the boundary operator content is given by

$$n_{\tilde{h}}{}^{h} = 1 \tag{152}$$
$$n^{0}_{\tilde{\frac{1}{16}},\tilde{\frac{1}{16}}} = n^{1/2}_{\tilde{\frac{1}{16}},\tilde{\frac{1}{16}}} = n^{1/16}_{\tilde{\frac{1}{2}},\tilde{\frac{1}{16}}} = 1 \tag{153}$$

The boundary condition changing operators are identified as follows:

1. ϕ_{+-} which produces a transition from $(+)$ to $(-)$ identified with $\phi_{\tilde{0}\tilde{\frac{1}{2}}}$. It transforms under the representation of the Virasoro algebra with weight 1/2.
2. ϕ_{+f} which produces a transition from $(+)$ to the free state is identified with $\phi_{(1,2)} = \phi_{(2,2)}$.

These results are consistent with what we found in Section 6.3.

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